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PLANE TRIGONOMETRY

FOR THE

USE OF COLLEGES AND SCHOOLS.

WITH EXAMPLES, PROBLEMS AND TABLES.

BY

I. J. BIRCHARD, M.A., PH.D.,

Mathematical Master, Collegiate Institute, Brantford.

Joint Author of "THE HIGH SCHOOL ALGEBRA," Part I. and Part II.

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PREFACE.

IN the following work on Trigonometry an attempt has been made to supply the Canadian student with a text-book adapted to the requirements of Canadian schools. In examining the available works on this subject, the writer has found them to consist of two classes. The one class consists of large heavy treatises, filled with difficult problems and references to the Differential and Integral Calculus, which render them unsuitable for elementary instruction. The other consists of works which simplify the subject by the easy process of omitting all the difficulties, and are, therefore, insufficient for laying a sure foundation for advanced work. Between these extremes it should be possible to unite thoroughness and accuracy with simplicity and brevity, whilst keeping the whole well within the comprehension of the average intelligent student. To fulfil these conditions is the object of the present volume.

Trigonometry constitutes a very important part of a student's mathematical course. The study of mathematics embraces two chief divisions, viz., Algebra and Geometry. The former treats of numerical relations, and deals with symbols. The latter treats of relations of form, and deals with objective magnitudes. Trigonometry unites the two. The meaning of the various symbols of the former, and the operations to which they are subjected, are exemplified by the diagrams of the latter, whilst the properties of the diagrams themselves are inferred from operations with symbols. The different classes of symbols, those of quantity, of operation, and of function, are distinguished from one another; ideas of limits are developed; the meaning of the infinitely great and the infinitely small is exemplified, the principle of continuity is illustrated, thus bringing into prominence the whole of the fundamental principles of mathematics.

To secure the desirable results just enumerated, it is necessary to study the two departments, the symbolical and the geometrical simultaneously, and to make constant reference from the one to the other. For this purpose the "line definitions" of the sine, cosine,

etc., have been given in addition to the usual "ratio" definitions. The values of the ratios for a considerable number of angles have been deduced geometrically. Various formulæ relating to triangles, polygons, the ratios of compound angles, etc., are deduced both from symbols and diagrams.

The theory will be found complete. The definitions and demonstrations are perfectly general, though, in some instances, particular cases are treated first to prepare the way for general investigations, and also to furnish material for exercises at each stage of the work. The demonstrations are given in the most convenient and concise form, and free from any mere explanatory matter. Where such is required it is given in separate paragraphs. This will be found a great convenience in preparing for examination.

The examples are numerous, and illustrate every portion of the work. They have been selected and arranged with care, and consist of those which are necessary for future use, and such as have been proved to be profitable exercise by the practical test of the class-room. Tedious and complicated problems have not been inserted.

A peculiar feature of the work is the insertion in an Appendix of a brief set of Mathematical Tables including the logarithms of numbers, the natural trigonometrical ratios and the logarithmic ratios calculated to five decimal places. These enable a student to choose his own method of solving a problem, to compare different modes of solution, and generally to obtain a grasp of the subject which is impossible without their use. Considerable use has been made of the natural functions, whose use is more readily understood than that of the logarithmic functions. The accuracy obtainable by the use of five-place tables is sufficient to illustrate all parts of the theory, and more than sufficient for all ordinary practical measurements.

In the preparation of the work, originality has not been attempted. The constant aim has been to put the matter in a teachable form, and to adapt it to the wants of the student.

The author thankfully acknowledges assistance from Mr. G. I. Riddell, B.A., Mathematical Master, Parkdale Collegiate Institute, Toronto, in reading the proof and verifying examples. From the great care taken to secure accuracy, it is believed that the errors have been reduced to a minimum, and that no serious difficulty will be experienced from that cause.

I. J. BIRCHARD.

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PLANE TRIGONOMETRY.

CHAPTER I.

DEFINITIONS AND FUNDAMENTAL CONCEPTIONS.

1. Plane Trigonometry in its primary meaning, is the science which treats of the relations existing between the lengths of the sides and the magnitude of the angles which form a plane triangle. In its more extended signification it treats of angular magnitude in general. Its subject-matter is **Geometry,** but the methods of investigation employed are chiefly those of **Algebra.**

THE TRIGONOMETRICAL LINE.

2. In pure Geometry a straight line is considered only as the distance between two fixed points. In Trigonometry we frequently consider the mode in which the line has been formed as well as its magnitude.

3. When a point moves through any distance in a fixed direction it traces out (or generates) a straight line which is bounded by the initial and final positions of the moving point. When the point returns to its original position it generates a line of the same length as before, and in the same position, but in the opposite direction. These two lines are distinguished by representing the one by a *positive* number, and the other by the

same number taken *negatively*. The number represents the length of the line (*i.e.*, the amount of motion of the moving point), and the sign determines the direction in which it has been generated.

4. A fixed point in a straight line, or a line produced is called a **point of section** of that line; and the lines intercepted between the point of section and the extremities of the given line are called **segments**.

Sometimes two or more points of section are taken, thus dividing the line, or the line produced, into a number of portions, each of which may be called a segment.

5. Segments are either positive or negative, according as they are generated by a point moving in the positive or in the negative direction. In naming a segment the direction is indicated by the order of the letters designating it; thus the segment AB indicates that it is measured from A to B.

6. Two or more segments are added by placing the initial point of the second segment upon the final point of the first; the initial point of the third upon the final point of the second, and so on; the sum of all the segments is the line reaching from the initial point of the first segment to the final point of the last segment added.

Thus $AB + BC = AC$. $\quad AC + CD + DE = AE$.

$AB + BA = 0$. $\quad AD + DC + CE = AE$.

It will be observed that when a positive segment is added the final point moves forward, but when a negative segment is added it moves backward. *Any straight line is thus equal to the algebraic sum of all its segments.*

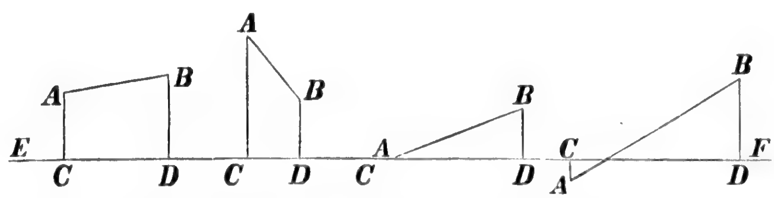
PROJECTIONS.

7. The projection of a point on a straight line is the foot of the perpendicular from the point on the line.

The projection of one straight line on another straight line is that portion of the second line intercepted between perpendiculars drawn to it from the extremities of the first line.

Thus the projection of the point A on EF is the point C, and the projection of the straight line AB on EF is the straight line CD.

The line to be projected must evidently be finite (*i.e.* of



limited length), whilst that upon which the projection is made must be considered unlimited.

8. The projection of a given line may be positive, zero, or negative. Thus, if F be in the positive direction from E, the projection of AB on EF is positive, whilst that of BA is negative, and if AB were drawn perpendicular to EF its projection would be zero.

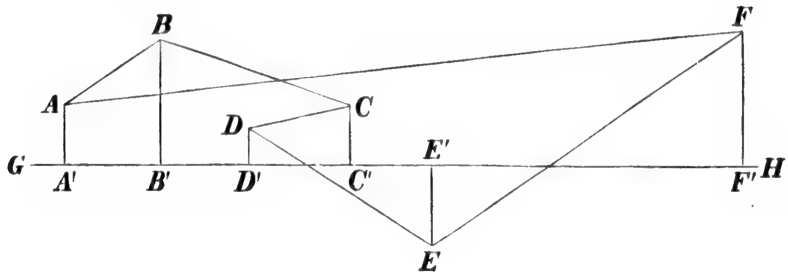
9. A broken line is a line made up of two or more finite straight lines with their extremities joined so as to form a continuous path but not a single straight line.

10. The algebraic sum of the projections of the several parts of a broken line upon any given straight line is equal to the like projection of the straight line joining the extremities of the broken line.

Let $ABCDEF$ be any broken line, GH any straight line, then the algebraic sum of the projections of AB, BC, CD, DE, EF is

$$A'B' + B'C' + C'D' + D'E' + E'F' = A'F'$$

and this is the same as the projection of the straight line AF which joins the initial and final points of the broken line.

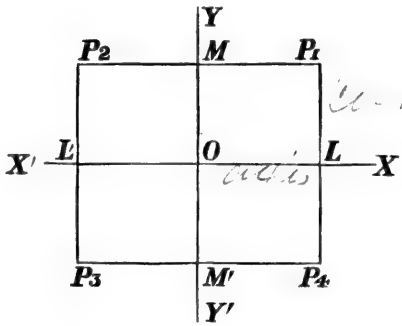


CO-ORDINATES OF A POINT.

11. Let $X'OX$ be any fixed horizontal line, $Y'OY$ a fixed vertical line, and from any point P in the same plane let PL and PM be drawn perpendicular to OX and OY, then OL and OM are the co-ordinates of the point P, with regard to OX and OY which are called **axes.** The position of P will be known when the *lengths* and *directions* of OL and OM are known, and these may be conveniently represented by algebraic numbers. The axes may be distinguished as the X-axis and the Y-axis, and the co-ordinates measured upon them may be called the x-co-ordinate and the y-co-ordinate respectively. A point is designated by naming its co-ordinates, the x-co-ordinate being always placed first.

Lines drawn horizontally to the right, or vertically upwards are usually considered positive, and those drawn in an opposite direction, negative. In the annexed diagram lines drawn in the direction OX or OY are positive, those in direction OX' or OY'

are negative. If the *lengths* of OL and OL' be each a, and of OM and OM' be each b, the co-ordinates of P_1 are a, b, those of P_2 are $-a, b$, those of P_3 are $-a, -b$, and those of P_4 are $a, -b$. For convenience of reference the points P_1, P_2, P_3, P_4, may be said to lie in the first, second, third, and fourth quadrants respectively.



EXERCISE I.

1. If A, B, C, D, E, are points in order in a straight line, name the segments of AD, BC, BD, giving three pairs for each.

2. In the previous example name the different lines of which AD may be considered a segment, name the remaining segment and state, in each case, whether it is positive or negative.

3. ABC is a triangle; show that the sum of the projections of AB and BC upon any straight line is equal to the projection of AC upon the same line.

4. The sum of the projections of the sides of a triangle, taken in order, upon any fixed straight line, is zero.

5. Generalize the two preceding theorems.

6. Draw a diagram showing the position of the points whose co-ordinates are, 2, 3; 3, 2; 2, -3; -2, 3; -3, -5; 4, 0; 0, -3; 0, 0.

7. Find the distance from the origin to each of the given points in the preceding example.

8. Show that the first four points in Ex. 6 all lie on the circumference of a circle, that the next two points are without this circle, the next within, and that the last is the centre.

9. The square of the distance between any two points on a plane is the sum of the squares of its projections upon any two straight lines at right angles to each other.

10. How is the position of a point affected by changing the signs of its co-ordinates? How is its distance from the origin affected?

11. A straight line, OP, revolves in a circle about O, name the quadrants in which (1) the x co-ordinate of P is positive, (2) the y co-ordinate is positive, (3) the quotient, $\frac{x}{y}$, of the co-ordinates is positive.

12. Two equal straight lines, OP and OQ, at right angles to each other revolve in the same direction in a circle starting from the initial lines of the first and second quadrants respectively. Show that, if at any time, the co-ordinates of P are a, b, the co-ordinates of Q are $-b$, a.

13. A straight line revolves about its centre, what relation exists between the co-ordinates of the two ends? If the arms are of different lengths, what relation exists?

14. Two equal straight lines, OP and OQ, revolve through equal angles in opposite directions, starting respectively from the initial and final lines of the first quadrant; show that if, at any time, a, b are the co-ordinates of P, then b, a are the co-ordinates of Q.

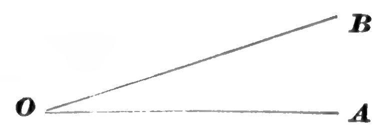
THE TRIGONOMETRICAL ANGLE.

12. In pure Geometry an angle is considered only as the inclination between two fixed straight lines which intersect. In Trigonometry we frequently consider the mode in which the angle has been formed as well as its magnitude.

13. When a straight line moves in a fixed plane about one of

its extremities as a fixed point it generates an angle which is bounded by the initial and final positions of the moving line. When the line is turned backwards to its original position it generates an angle of the same magnitude as before and in the same position, but in the opposite direction. These two angles are distinguished by representing the one by a positive number and the other by the same number taken negatively. The number represents the magnitude of the angle (*i.e.*, the amount of angular motion of the moving line) and the sign determines the direction in which it has been generated.

14. In naming an angle the direction in which it has been described is indicated by the order of the letters; thus the angle AOB indicates that the generating line has moved from OA to OB. The angle BOA is the negative of the angle AOB.

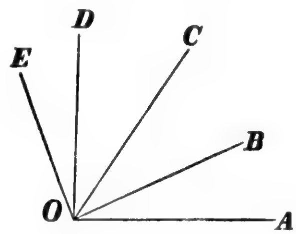


15. It is a matter of indifference which direction we choose as the positive direction of rotation, but when the choice is once made, it must be adhered to throughout any one investigation. Mathematicians are, however, unanimous in deciding that *the positive direction of rotation shall be contrary to that of the hands of a watch*, which decision will be followed throughout this volume. The lines which bound an angle, measured from the angular point outward, are always considered positive. When, therefore, the positive direction of any one line has been arbitrarily chosen, the positive direction of a perpendicular to it may be determined by turning the original line through one right angle in the direction of rotation chosen as positive. Thus, the positive direction of the axis of Y is obtained by turning the axis of X through a positive right angle. (Art. 11.)

16. Two or more angles may be added by placing the initial line of the second angle on the final line of the first, the initial line of the third upon the final line of the second, etc.; the angle

contained by the initial line of the first angle and the final line of the last will be the sum of all the angles.

17. Let a line revolve around the point O from OA to OE through OB, OC and OD, then, since this is considered the positive direction of rotation, the angles AOB, AOC, BOE, etc., are positive, whilst BOA, COA, EOB, etc, are negative.



Also $AOB + BOC = AOC$. $AOC + COD + DOE = AOE$.
$AOB + BOA = 0$. $AOC + COB + BOD = AOD$, etc.

It will be observed that when a positive angle is added the revolving line moves forward, but when a negative angle is added it moves backward.

18. In pure Geometry the angular magnitudes considered are usually each less than two right angles, but it is evident that there is no limit to the angular motion of the revolving line, and consequently there is no limit to the angular magnitude which may be described. An angle may be infinite in magnitude, the same as a line may be infinite in length, the former being generated by the continuous revolution of a line and the latter by the continuous motion of a point.

19. At the end of each complete revolution the revolving line will coincide with the initial line and the *geometrical* angle between them vanishes. At any other point in the course the geometrical angle will be the same in the successive revolutions, but the *trigonometrical* angle is assumed to include the complete revolutions as well as the fractional part of one revolution.

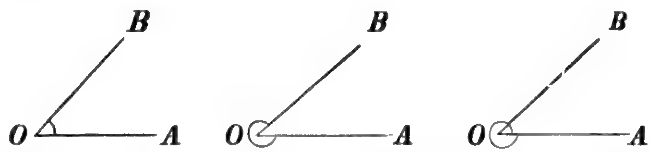
20. Two geometrical angles are equal when the lines which bound one angle may be made to coincide with those which bound the other (as in Euc. I., 4), but this is not necessarily true of trigonometrical angles, for the following reasons:

(1) Either line may be selected as the initial line.

(2) When the initial line has been selected, the revolving line may turn in either of two directions.

(3) The revolving line may make any number of complete revolutions before coming to rest.

Thus, the three angles in the accompanying figure are geometrically equal, but the amount of rotation, as indicated by the spirals, by which they have been severally generated, is in each case different.



Moreover each angle may be generated by starting from either OA or OB. Six different trigonometrical angles are thus represented by the same bounding lines, and by varying the number of complete revolutions the number of angles represented may evidently be increased to any extent.

21. The motion of the hands of a clock illustrates this part of the subject. Each hand, by its revolution, describes an angle by revolving about the centre of the dial. The initial line is that joining the centre of the dial to the mark for the hour of twelve. The minute hand marks the geometrical angle between its position at a given time and the initial line, and thus determines the number of minutes *since the beginning* of the hour or *before its completion.* The hour hand marks the number of complete revolutions which the minute hand has made since twelve o'clock, and the two hands combined mark the trigonometrical angle described by the minute hand during this time.

22. The quarter part of a complete revolution is called a **quadrant,** and corresponds to the geometrical right angle. In describing a complete revolution in the positive direction the quadrants are called first, second, third and fourth, respectively, in the order in which the revolving line passes through them, and an angle is said to lie in a particular quadrant when the revolving line, after describing the angle, comes to rest in that quadrant.

MEASUREMENT OF ANGLES.

23. To measure an angle we select a fixed angle as the unit of measure, determine how often this unit must be repeated to produce the given angle, and to this number prefix the sign $+$ or $-$, according as the angle has been generated in the positive or the negative direction.

24. The natural unit of angular magnitudes is a complete revolution, but this being too large for convenience, smaller units are derived from it in one of two distinct ways:

(1) By dividing a complete revolution into an exact number of equal parts.

(2) By measuring off arcs of a given length on the circumference of a circle of given radius.

The former method is the more convenient for practical work, and the latter for theoretical investigations.

25. A complete revolution divided into four equal parts determines the right angle, which is the basis of the ordinary system of measurement. The right angle is divided into 90 equal parts called degrees, the degree into 60 equal parts called minutes, and the minutes into 60 equal parts called seconds. We have, then, the following table:

1 revolution	=	4 right angles		
1 right angle	=	90 degrees,	written	90°
1 degree	=	60 minutes,	"	$60'$
1 minute	=	60 seconds,	"	$60''$

From the division of the degree and the minute each into *sixty* equal parts, the above is sometimes known as the *Sexagesimal Method;* it is also known as the English Method.

26. Another mode of subdividing the right angle, which is theoretically more convenient than the former, is given in the following table:

1 right angle	=	100 grades,	written	100^g
1 grade	=	100 minutes,	"	$100'$
1 minute	=	100 seconds,	"	$100''$

From the subdivisions being made into *hundreds* in each case, this is known as the *Centesimal Method.*

27. The Centesimal Method was proposed by a number of French mathematicians at the beginning of the present century. Its theoretical superiority is at once evident, but there are many practical difficulties which prevent its adoption. The results of many valuable observations, especially in astronomy, have been recorded, valuable mathematical instruments graduated, and trigonometrical tables calculated, all according to the Sexagesimal Method. The labor and expense involved in a change of system would be greater than the resulting gain; hence the only purpose this method serves is that of furnishing exercises in reduction for the student of Trigonometry. Other methods of subdivision are also employed for special purposes. Thus, for reckoning time, we divide a complete revolution of the earth upon its axis into 24 equal parts, and call the time occupied in describing one of those parts an hour. Astronomers divide the ecliptic into 12 equal parts, called Signs of the Zodiac; mariners divide the horizon into 32 equal parts, called Points of the Compass, etc. It will be readily perceived that every system of angular measurement must be ultimately based upon the circle considered as one complete revolution.

28. When two angles taken together make up one right angle each is said to be the **complement** of the other, and when they together make two right angles each is said to be the **supple-**

ment of the other. In ordinary geometry complementary angles must each be less than a right angle, and supplementary angles must each be less than two right angles, but these restrictions are not observed in trigonometry. If A denote any angle whatsoever, then $90° - A$ is its complement, and $180° - A$ is its supplement.

29. The student will find but little difficulty in performing the various reductions required. The following are a few easy examples:

Ex. 1. Express 13° 10′ 15″ as the decimal of a right angle.

60	15.000	seconds.
60	10.25	minutes.
90	$13.1708\dot{3}$	degrees.
Result	.14634259 . . .	of a right angle.

Ex. 2. Express $43^g\ 15^{\backprime}\ 65^{\backprime\backprime}$ in degrees, etc.

$$43^g\ 15^{\backprime}\ 65^{\backprime\backprime} = .431565 \text{ of a right angle.}$$

$$\begin{array}{rl} 90 & \\ \hline 38.840850 & \text{degrees.} \\ 60 & \\ \hline 50.451000 & \text{minutes.} \\ 60 & \\ \hline 27.060000 & \text{seconds.} \end{array}$$

Result, 38° 50′ 27″.06

Ex. 3. Find the angles of an isosceles triangle, providing the number of degrees in one of the base angles is $\frac{3}{10}$ of the number of grades in its supplement.

Let x denote the number of degrees in a base angle.

Then $\frac{10}{9}(180 - x)$ is the number of grades in its supplement.

Therefore $\quad x = \frac{3}{10} \text{ of } \frac{10}{9}(180 - x)$

From which $x = 45$.

The triangle is, therefore, a right-angled isosceles triangle.

EXERCISE II.

1. Draw the boundary lines of the following angles and indicate the directions in which they have been described:

3 right angles.	5 right angles.	– 3 right angles.
4 right angles.	– 4 right angles.	125 right angles.
750 degrees.	– 2850 degrees.	1000 grades.

2. If n denotes any integer, positive or negative, draw the angles represented by $\left(2n+\frac{1}{2}\right)$ right angles, and $\left(4n+\frac{2}{3}\right)$ right angles.

3. The geometrical angle, AOB, contains 30°; write down a general expression for all the different trigonometrical angles contained by the same bounding lines.

4. Express the following angles as the decimal of a right angle:

(1) 23° 17′ 14″. (2) 127° 15′ 25″. (3) 37^g 14` 83``.

5. Change the following decimals of a right angle into degrees, minutes and seconds; also into grades, minutes and seconds:

(1) .07625. (2) 1.234506. (3) 3.0125. (4) .00075.

6. Find the complement and the supplement of each angle in the preceding example. Draw a diagram representing the various angles, and distinguish between the positive and the negative angles which occur.

7. How many degrees in the angle between the hands of a clock at 2 o'clock? At half-past two?

8. How many degrees in an angle of an equilaterial triangle? How many grades in an exterior angle?

9. If the unit of angular measurement be $\frac{1}{12}$ of a right angle, what number will represent the angle of a regular hexagon?

10. The angle of a regular pentagon is represented by 24, how many degrees in the unit of angular measurement?

11. If D be the number of degrees in an angle, and G the number of grades in it, then $\frac{D}{9} = \frac{G}{10}$. $\frac{D}{10} = \frac{G}{9}$

12. Divide a right angle into two parts such that the number of degrees in one part is one-tenth the number of grades in the other part.

13. The difference between two angles is $2°$, and the number of grades in the second is greater by 5 than the number of degrees in the first; find the angles.

14. Find the times between 11 and 12 o'clock when the angle between the two hands is 110 degrees.

15. Through how many degrees does the minute hand of a clock move in one minute of time?

16. Through how many English minutes of angular measure does the earth turn on its axis in one minute of time?

17. The moon revolves around the earth in 29 days and 2 hours; what is its average angular velocity per hour?

18. What trigonometrical angle does the minute hand of a clock describe between midnight and (1) 3.15 a.m.; (2) 4.50 p m.?

19. The longitude of the Paris observatory is $2° 20' 9.45''$ east; that of the observatory at Pulkowa is $30° 19' 39.9''$ east; what is the time at Paris when it is $1^h 5^m 12^s$ A.M. of Sept. 3rd at Pulkowa?

20. The circumference of a wheel is 10 feet; through what angle will it turn in moving over a space $3\frac{1}{2}$ feet? If the wheel move forward at the rate of 12 miles per hour what is the angular velocity of the wheel per minute, taking a right angle as the unit of measurement?

21. The front wheel of a bicycle is 15 feet in circumference and makes 100 revolutions per minute in passing around a cir-

cular course half a mile in circumference; through how many degrees will the rider move in 11 seconds?

22. Two cog-wheels, whose circumferences are 20 and 15 inches respectively, work together. Through how many inches must a point in the circumference of the larger wheel move so that diameters of the wheels, formerly parallel, may now be at right angles?

23. The angles of a triangle are in arithmetical progression, and the number of grades in the least is two-thirds of the number of degrees in the greatest; find the angles.

24. $ABCD$ is a quadrilateral inscribed in a circle. The angles A and B together contain 156°, and the number of grades in B is greater by 15 than the number of degrees in A; find all the angles.

25. CAB is an isosceles triangle, C being a right angle; show that the angle between the *lines* AB and BC is one-third of the angle between the *directions* AB and BC.

26. ABC is a triangle. If a straight line be made to revolve from the initial position BC through the angles B, A, and C, in succession, show that its direction will be reversed, and that consequently the sum of the angles of any triangle are together equal to two right angles.

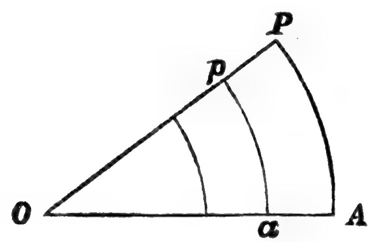
27. ABC is a triangle, the angle B being greater than the angle C; the side AB is produced to D, and AE is drawn to meet the base in E, making the angle CAE equal to the angle ACE; the angle B is equal to two-thirds of the sum of CBD and BAE, and to 4 times BAE alone; find the angles of the triangle.

$\angle B = \frac{2}{3}\{(180-B)+(A-C)\}$

CHAPTER II.

ANGLES MEASURED BY CIRCULAR ARCS.

30. When an angle is generated by a straight line turning about a fixed point, every point in the line describes the arc of a circle. These arcs are all of different lengths, but each is evidently the same fraction of a complete circle that the angle



is of a complete revolution. For this reason *arcs* and *angles* are frequently interchangeable terms. If the angle AOP contains 30 degrees, then each of the arcs AP, ap, also contains 30 degrees, a degree in either case being simply a certain fraction of a complete revolution. But if we wish to connect the angle with the *length* of an arc we must, then, also consider the length of the corresponding radius. For this purpose we must prove that the ratio of the circumference of a circle to its diameter is the same for all circles. In elementary work this theorem is usually assumed without proof, and the student may, if necessary, adopt that course in the present instance, reserving the difficulty for future consideration.

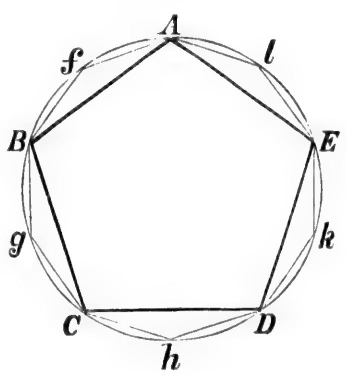
31. We shall assume the truth of the two following statements :—

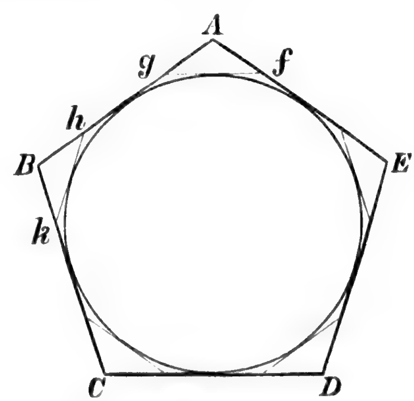
1. The perimeter of a regular polygon inscribed in a circle is less than the circumference of the circle.

2. The perimeter of a regular polygon described about a circle is greater than the circumference of the circle.

The truth of these statements will be evident upon reflection, though neither is capable of proof in the strict sense of the word. Anything which can be offered as a proof will be found, upon strict examination, to contain assumptions quite as difficult as the propositions to be proved.

32. Let the regular polygon *ABCDE* in Fig. 1 be inscribed in, and in Fig. 2 described about, a circle, then the perimeter of

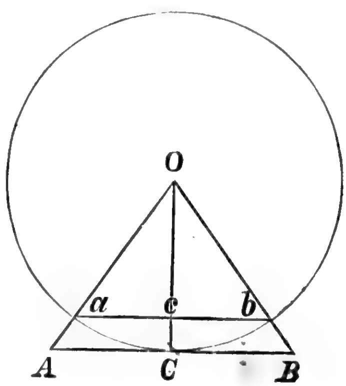




the former is less than the circumference of the circle, but the perimeter of the latter is greater. If each of the arcs be bisected and the number of the sides of each polygon doubled, the perimeter of the inscribed polygon will be increased and that of the circumscribed polygon will be diminished. For in Fig. 1 *Af* and *fB* are together greater than *AB*; *Bg* and *gC* greater than *BC*, etc.; and in Fig. 2, *fA* and *Ag* are together greater than *fg*, etc., from which the required results easily follow. In

the next article we shall prove that by sufficiently increasing the number of sides the perimeters of the two polygons may be made as nearly equal as we please, and since the circumference of the circle is greater than the one and less than the other, its value may be found to any required degree of accuracy by finding the perimeter of a polygon of a sufficient number of sides.

33. *The circumference of a circle is the limit of the perimeter of either the inscribed or the circumscribed regular polygon when the number of sides of the polygon is indefinitely increased.*



Let ACB be a side of a regular polygon cicumscribed about a circle, ab the side of the corresponding inscribed polygon, let P and p denote the perimeters of the polygons, and let the number of sides of each be n. From the similar triangles OAC, Oac, we have from Euc. VI., 4,

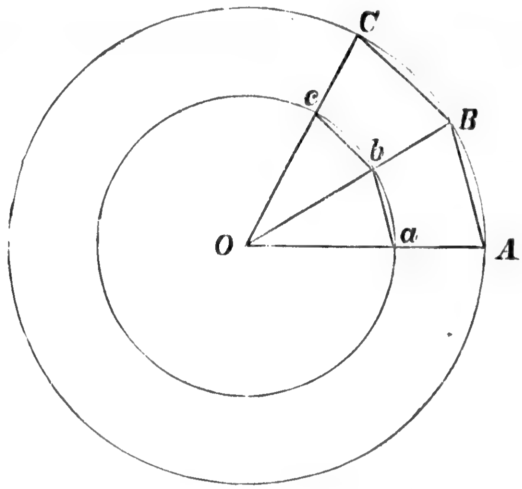
$$\frac{OA}{Oa}=\frac{AC}{ac}=\frac{2n.AC}{2n.ac}=\frac{P}{p}.$$

Now, by sufficiently increasing n, the length of a side may be made indefinitely small, and, therefore, since the point A may be brought indefinitely near to C, OA may be made as nearly equal as we please to OC or Oa, and consequently P may be made as nearly equal as we please to p. But the circumference

of the circle lies between P and p, therefore, it becomes ultimately equal to each of them.

34. *The ratio of the circumference of a circle to its diameter is a fixed number;* i.e., *it is the same for all circles.*

Let $ABC\ldots.$, $abc\ldots.$, be any two circles; place them so that they have the same centre O; let $A, B, C\ldots.$ be the angular points of a regular polygon of n sides inscribed in the outer



circle. Join $OA, OB, OC\ldots.$, meeting the inner circle in $abc\ldots.$, then $a, b, c\ldots.$, will be the angular points of a similar polygon inscribed in the smaller circle. Let P and p denote the perimeters of the polygons, D and d their diameters; then from the similar triangles OAB, Oab, we have

$$\frac{OA}{Oa}=\frac{AB}{ab}=\frac{n.AB}{n.ab}=\frac{P}{p}=\frac{C}{c}$$

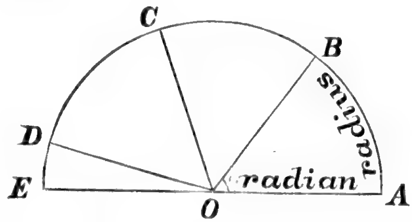
the last equality becoming true when n is made indefinitely great.

And since $D = 2OA$, $d = 2Oa$, we have

$$\frac{C}{c} = \frac{D}{d} \text{ or } \frac{C}{D} = \frac{c}{d}.$$

which proves the proposition.

35. The value of the ratio of the circumference of a circle to its diameter is an incommensurable number; *i.e.*, its value cannot be expressed either as a vulgar fraction or a decimal. And since we have no convenient numerical expression for its exact value, it is customary to denote it by π, the initial letter of the Greek word denoting circumference. Its value to 10 places of decimals is $\pi = 3.1415926536$. Approximate values frequently used are $3\frac{1}{7}$, $\frac{355}{113}$, and 3.1416.



36. A **radian** is the angle subtended at the centre of a circle by an arc equal to the radius. Let the arc, AB, of the circle whose centre is O, be equal in length to the radius AO, then the angle AOB is a radian.

37. *All radians are equal to one another.*

Since 2π times the radius of a circle equals the circumference, therefore π times AB = semicircumference

or $\qquad$ arc $AB = \dfrac{1}{2\pi}$ of circumference

therefore $\qquad \angle AOB = \dfrac{1}{2\pi}$ of a complete revolution $\quad$ Euc. VI. 33.

$= \dfrac{1}{2\pi}$ of 4 right angles

$= \dfrac{2}{\pi}$ of a right angle.

Hence a radian is a fixed fraction of a right angle, and consequently does not depend upon the magnitude of the circle.

Cor. π radians make two right angles, and one right angle is equivalent to $\frac{\pi}{2}$ radians.

38. The circular measure of an angle is the number which denotes the angle when the unit is a radian; or,

The circular measure of an angle is the number of radians which it contains.

39. The circular measure of angles is usually denoted by a Greek letter, and the number of degrees by an English letter. In either case the letter alone denotes a number, and only in connection with the angular unit does it denote an angle. Thus, θ denotes a number, just as letters in algebra denote numbers, but we frequently speak of "the angle θ," and, if so, we mean θ radians. The letter π is restricted to the particular number 3.1415 . . . ., which is the number of radians in two right angles. Radians may be specially marked by ^{r}, thus, θ^r, 2^r, etc., but this is seldom necessary. Of course, the ^{r} used in this way must not be confused with an exponent.

40. *To find the circular measure of the angle at the centre of a circle subtended by an arc of given length.*

Let AOC be the given angle, l the length of the arc AC, r the radius of the circle, and θ the circular measure required.

Then $$\theta = \frac{\angle AOC}{\angle AOB} \qquad \text{Fig. of Art. 36.}$$

$$= \frac{\text{arc } AC}{\text{arc } AB} \qquad \text{Euc. VI. 33.}$$

$$= \frac{l}{r}$$

This result should be memorized, both in the form just given and also in the equivalent form $l = r\theta$.

Ex. 1. The angle at the centre of a circle 12 feet in diameter subtended by an arc 2 feet long is $\frac{1}{3}$ of a radian.

Ex. 2. The length of the arc, radius 3 feet 2 in., subtending an angle whose circular measure is 1.235 is $1.235 \times 38 = 46.930$ inches.

41. *To find the circular measure of an angle of a given number of degrees or grades; and the converse.*

Let θ be the circular measure of the angle, n the number of degrees in it, g the number of grades; then

since θ radians $= n$ degrees $= g$ grades
and π " $= 180$ " $= 200$ "

Therefore $$\frac{\theta}{\pi} = \frac{n}{180} = \frac{g}{200}$$

from which, if any one of θ, n, g, be given, the other two may be found.

Ex. Find the circular measure of $2^\circ\ 37'\ 25''$.

$$\begin{aligned} 2^\circ\ 37'\ 25'' &= 2.62361 \text{ degrees} \\ &= 2.62361 \times \frac{\pi}{180} \text{ radians} \\ &= .04579 \text{ radians.} \end{aligned}$$

42. Problems in trigonometry are solved by the same general principles as problems in algebra. Unknown quantities are represented by letters, the conditions of the problem are expressed by equations, and the equations are solved in the usual way. The following is a simple example:

Ex. Find the angle between two tangents to a circle 10 feet in diameter, providing the points of contact divide the circumference into arcs, one of which is 6 inches longer than the other.

Let AP, AQ, be the tangents, O, the centre of the circle, θ, the circular measure of the angle A. Then, since the angles at P and Q are right, the angles at A and O are supplementary.

Therefore, the circular measure of the *obtuse* angle POQ is $\pi - \theta$, and that of the *reflex* angle POQ is $2\pi - (\pi - \theta)$ or $\pi + \theta$.

And since the radius of the circle is 5 feet, the lengths of these arcs in feet are $5(\pi - \theta)$ and $5(\pi + \theta)$.

Therefore, by conditions of the problem,

$$5(\pi + \theta) - 5(\pi - \theta) = \frac{1}{2}$$

from which

$$\theta = \frac{1}{20}$$

The angle is, therefore, $\frac{1}{20}$ of a radian, which, expressed in degrees, is $\frac{1}{20}$ of $\frac{180^\circ}{\pi} = 2^\circ.8647889 = 2^\circ\ 51'\ 53''$ nearly.

EXERCISE III.

1. Express the following angles in circular measure :

(1) 30°. (2) 45°. (3) $1^\circ\ 15'$. (4) π°.

(5) $57^\circ.295\ldots$ (6) $2''.03$. (7) $75^g\ 83'\ 25''$. (8) $\frac{100^g}{\pi}$.

2. Change the following angles from radians to degrees :

(1) π. (2) $\frac{\pi}{3}$. (3) 3.1416. (4) θ.

(5) $\left(\frac{\pi}{3}\right)^2$ (6) $\frac{1}{\pi}$. (7) 2.0425. (8) $\sqrt{\pi}$.

3. The radius of a circle being 10 feet, find the length of the arc subtending an angle at the centre of

(1) $\frac{\pi}{12}$. (2) 75°. (3) $1'\ 12''.5$. (4) $63^g\ 15'\ 95''$.

4. Find the radius of the circle when an arc 10 feet in length subtends the following angles :

(1) $\frac{2\pi}{3}$. (2) 1°. (3) 2.345 right angles. (4) a radian.

1 radian = $57^\circ.295$

5. Find in degrees the angles at the centre of a circle of 5 yards radius subtended by arcs whose lengths are

(1) 10 in. (2) 2 feet 3 in. (3) 30 yards. (4) π yards.

6. Express in each measurement the angle through which a wheel 5 feet in diameter will turn in moving forward 10 feet.

7. From a regular hexagon inscribed in a circle show that the circumference is more than three times the diameter and that a radian is less than 60°.

8. A point moves in a circle through an angle of one minute in a second of time; find the time of a complete revolution.

9. Find the circular measure of an angle of a regular polygon of 8, 12, 20, n, sides.

10. The vertical angle of an isosceles triangle contains $1\frac{1}{2}$ radians; how many degrees in one of the base angles?

11. Find the circular measure of the angle between the hands of a clock at ten minutes past three.

12. If the unit of angular measure be the angle subtended at the centre of a circle by the side of an inscribed regular hexagon, what number will denote the angle between the hands of a clock at twenty minutes after three?

13. Show that the number of degrees in an exterior angle of a regular nonagon is equal to the number of grades in that of a regular decagon.

14. The angles of a triangle are in arithmetical progression and the number of degrees in the least is to the circular measure of the greatest as $60:\pi$; find the angles.

15. The number of sides in one regular polygon is to the number in another as $2:3$, and an angle of the former is to one of the latter as $21:22$; find the number of sides in each.

16. On a circle of 75 feet radius an angle of 15° subtends an arc of 19 feet 7.61944 in.; find the value of π.

17. Find the distance between two places situated on the

same meridian, the differences of whose latitudes is $1° 11' 13''.5$, the radius of the earth being 4,000 miles.

18. If the diameter of the moon subtend an angle of $30'$ at the eye of an observer, and the diameter of the sun an angle of $32'$, and if the distance of the sun be 375 times the distance of the moon, find the ratio of the diameter of the sun to that of the moon.

19. On the 31st of December the sun subtends an angle of $32' 36''$, and on 1st of July an angle of $31' 32''$; find the ratio of the distances of the sun from the observer on those two days.

20. The three angles of a triangle are denoted by the same number when expressed in degrees, grades, and radians respectively. Find this number, and show approximately the form of the triangle.

21. A triangle ABC is inscribed in a circle, and the arcs BC, CA, AB, have lengths a, b, c, respectively; find the circular measure of the angles of the triangles.

22. The circumference of one circle is just long enough to subtend an angle of one radian at the centre of another; how many degrees in the arc of the larger circle whose length equals the arc of a radian in the smaller circle?

23. Two circles touch the base of an isosceles triangle at its middle point, one having its centre at, and the other passing through, the vertex. If the arc of the greater circle included within the triangle be equal to the arc of the lesser circle without the triangle, find the vertical angle of the triangle.

24. Two circles are described from the vertex of an isosceles triangle as centre, the one touching the base, and the other passing through the extremities of the base, and the difference of the intercepted arcs is $\frac{11}{21}$ of the difference of their radii; find the vertical angle of the triangle ($\pi = 3\frac{1}{7}$).

25. Three circles, whose radii are a, b, c, touch each other ex-

ternally and their centres are joined; the intercepted arcs are all equal; find the circular measure of the angles of the triangle.

26. The exterior angle of a triangle contains twice as many grades as one of the interior opposite angles contains degrees, and the sum of these two angles is $4\frac{2}{3}$ radians; find the number of degrees in the remaining interior opposite angle $(\pi = 3\frac{1}{7})$.

27. The sides of an irregular pentagon inscribed in a circle subtend angles at the centre in arithmetical progression; the largest angle is five times the least and the largest side is a, find the length of the arc this side subtends.

CHAPTER III.

THE TRIGONOMETRICAL RATIOS.

43. Let the straight line OP revolving from OX, describe any angle XOP; take OX, and OY making a positive right angle

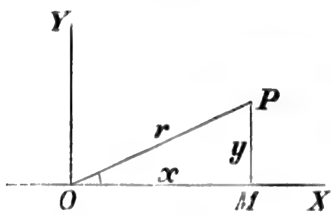


Fig. 1.

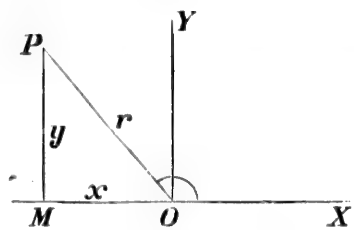


Fig. 2.

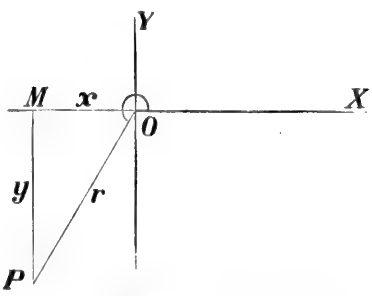


Fig. 3.

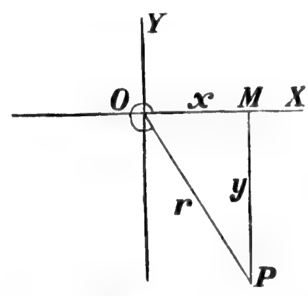


Fig. 4.

with OX, as axes of reference; let x, y be the co-ordinates of any point P in OP, r the distance of P from the centre O; then the six ratios which exist between the numbers x, y, r, are called

the **trigonometrical ratios** of the angle XOP. They are distinguished each by a separate name, as follows:

$$\frac{y}{r} = \textbf{sine of } XOP. \qquad \frac{r}{y} = \textbf{cosecant of } XOP.$$

$$\frac{x}{r} = \textbf{cosine of } XOP. \qquad \frac{r}{x} = \textbf{secant of } XOP.$$

$$\frac{y}{x} = \textbf{tangent of } XOP. \qquad \frac{x}{y} = \textbf{cotangent of } XOP.$$

The names of these ratios are usually abbreviated. Denoting the angle XOP by A, they are written thus:

$$\sin A,\ \cos A,\ \tan A,\ \operatorname{cosec} A,\ \sec A,\ \cot A.$$

In this connection it is customary to define the **versed sine**, and **coversed sine**, though these terms do not denote ratios. They are defined and written thus:

$$\operatorname{vers} A = 1 - \cos A, \quad \operatorname{covers} A = 1 - \sin A.$$

44. The direction of revolution is indicated as positive for each angle in the preceding Art., but if the line OP had revolved in the *opposite* direction, and had come to rest in the same position, the ratios of the *negative* angles thus described would in each case have been exactly the same. The ratios for any angle are the same as those for the remainder of a complete revolution *described in the opposite direction*.

45. The signs of the numbers denoted by x, y, r should be carefully noted.

1. r is always positive, since it denotes one of the bounding lines of the angle measured *outwards*.

2. x is positive when the angle lies in the first or the fourth quadrant; negative in the other two.

3. y is positive when the angle lies in the first or the second quadrant; negative in the other two.

For example, in Fig. 3, if the *lengths* of OM and MP are three

and four inches respectfully, then OP is five inches, and we have

$$\sin XOP = \frac{4}{5}, \cos XOP = \frac{-3}{5}, \tan XOP = \frac{-4}{-3} = \frac{4}{3}.$$

46. The student should carefully observe:

1. The trigonometrical ratios, being the ratios of the lengths of lines, *are abstract numbers*, and as such may be treated as ordinary algebraical quantities.

Thus $(\sin A) \times (\sin A) = (\sin A)^2$, which for convenience is usually written $\sin^2 A$.

2. The name of a ratio must never be separated from the angle to which it refers.

Thus we cannot assume $\sin 2A = 2 \sin A$, for this would be treating sin as a numerical factor.

Similar remarks apply to all the ratios.

47. The definitions of the trigonometrical ratios have been given in their most general form, to serve as a sure basis for all subsequent investigations. But the angles of a triangle, with which elementary trigonometry is chiefly concerned, are always considered positive and always less than two right angles. With this simplification, the ratios of any angle may be readily written according to the directions in the following Art.

48. In either of the lines bounding a given angle take any point, and from it draw a perpendicular to the remaining line (produced, if necessary), then

1. The sine of the angle is the ratio of the *opposite* side to the hypothenuse.

2. The cosine is the ratio of the *adjacent* side to the hypothenuse.

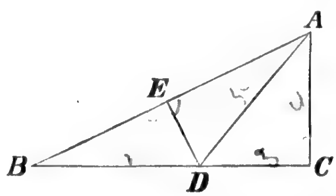
3. The tangent is the ratio of the opposite side to the adjacent side.

4. The cosecant, secant, and cotangent are the reciprocals respectively of the sine, cosine, and tangent.

In the above the perpendicular will always be positive, but the base will be positive for acute angles, and negative for obtuse angles.

49. The distinction between algebraical and geometrical magnitudes must be kept clearly in mind. In algebra the symbols stand for *numbers*, and the latter, together with the unit of measurement, represent the magnitude. In geometry, the symbols represent the magnitudes directly, without any reference to number, they are mere *names*, and consequently do not admit of algebraical operations being performed upon them. But when the parts of a diagram have once been designated by letters used in a geometrical sense, it is frequently convenient to use the same symbols when algebraical operations are involved. Thus, if AB and CD are two straight lines, we shall frequently use such expressions as $\frac{AB}{CD}$ to denote the ratio of the numbers which are the algebraical measures of the given lines.

50. Adopting this notation, we give a number of examples.



From the given diagram, in which the angles at C and E are right angles, we have

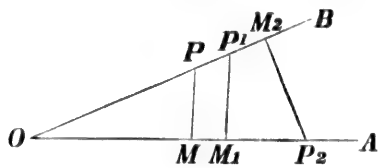
$$\sin B = \frac{CA}{BA} \text{ or } \frac{ED}{BD}, \quad \cos B = \frac{BC}{BA} \text{ or } \frac{BE}{BD}.$$

Sin $ADB = \frac{CA}{DA} = \sin ADC$, hence the sines of supplementary angles are equal.

Cos $ADC = \frac{DC}{DA}$ and cos $ADB = \frac{DC}{DA}$, yet the cosines of supplementary angles are *not* equal. For, with reference to the angle ADC, DC is positive, being one of the bounding lines, but with regard to the angle ADB it is negative, being one of the bounding lines DB, produced backwards. Hence cos $ADB = -$ cos ADC, and the cosines of supplementary angles are equal numerically but of opposite sign.

In the same way, many other relations between the ratios of complementary or supplementary angles may easily be deduced.

51. *The trigonometrical ratios for any given angle remain unchanged wherever the point be chosen from which the perpendicular is drawn.*



Let P, P_1, P_2 be different points in the bounding lines of the angle AOB, from which perpendiculars are drawn to the opposite side. Then the three triangles POM, P_1OM_1, P_2OM_2, have each a right angle, and the angle at O in common, therefore the third angles of each are equal and the triangles are similar.

Therefore $$\frac{MP}{OP} = \frac{M_1P_1}{OP_1} = \frac{M_2P_2}{OP_2}.$$ Euc. VI., 4.

Each of these fractions is by definition sin AOP which is, therefore, constant for all positions of P in each of the bounding lines. Similarly the constancy of each of the other ratios may be shown.

Cor. If the sines of two angles be equal, both being acute or both obtuse, the angles themselves are equal. The same is evidently true for each of the other ratios.

EXERCISE IV.

1. ABC is a triangle right-angled at C, and the sides opposite A and B are 3 and 4 respectively. Write down the trigonometrical ratios of the acute angles.

2. In the preceding example prove the following:

$\sin^2 A + \cos^2 A = 1$, $1 + \tan^2 A = \sec^2 A$, $1 + \cot^2 B = \operatorname{cosec}^2 B$.

3. In the figure of Art. 50, if $BD = 7$, $DC = 3$, $CA = 4$. Find the numerical value of $\tan B$, $\tan ADB$, $\tan ADC$, and $\dfrac{ED}{EB}$.

4. Draw an angle whose tangent is (1) $\frac{5}{12}$, (2) $-\frac{5}{12}$. Find the sine and the cosine of each of these angles.

5. $ABCD$ is a square, AC a diagonal, and the side DC is produced to E. Write down the numerical values of the following ratios:

(1) $\sin CAB$, (2) $\tan ACB$, (3) $\sec ACB$, (4) $\cos ACE$,
(5) $\tan ACD$, (6) $\cot ACE$, (7) $\sin ACE$, (8) $\cos ACB$.

6. From the vertex of an equilateral triangle a perpendicular is drawn to the base; find the ratio of its length to the length of a side. Of what angles is this ratio the sine and the cosine respectively?

7. Find the tangent of the angle which a vertical rod 6 inches long subtends at a horizontal distance of 3 feet. If the tangent of the angle subtended by a spire 300 feet distant be 1.125, find the height of the spire.

8. In the triangle ACB, C being a right angle, AD is drawn to meet the base in D, and $AB = 37$, $AC = 12$, $CB = 35$, $CD = 5$. Find the following trigonometrical ratios.

(1) $\sin ABC$, (2) $\sin ADC$, (3) $\cos ADC$, (4) $\cos ADB$,
(5) $\sin DAC$, (6) $\sin BAC$, (7) $\sin BAD$, (8) $\tan BAD$,
(9) $\tan ADB$, (10) $\cot ADC$, (11) $\sec ADB$, (12) $\operatorname{cosec} ADC$.

9. In a right-angled triangle ACB, write all the ratios of each

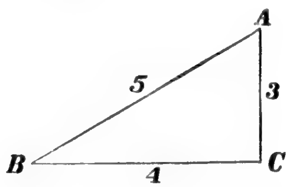
of the complementary acute angles A and B, and observe the relations which exist between them.

10. Write the ratios of two supplementary angles and state the relations which exist between them.

Relations Between the Ratios.

52. When one of the trigonometrical ratios of an angle is given, and also the quadrant in which the angle is situated, the other ratios may be found. We shall first give some simple examples and afterwards proceed to a more general investigation.

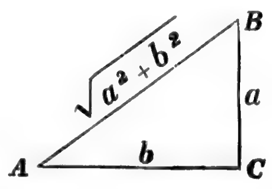
Ex. 1.—The sine of an acute angle is $\frac{3}{5}$. Find the other trigonometrical ratios of the angle.



Let ABC be a right-angled triangle, and let the sides AC and AB be respectively 3 and 5 units in length, then the angle B is the angle required. From the values given $BC = \sqrt{5^2 - 3^2} = 4$.

Then $$\cos B = \tfrac{4}{5},\ \tan B = \tfrac{3}{4},$$
$$\sec B = \tfrac{5}{4},\ \cot B = \tfrac{4}{3}, \text{ etc.}$$

Ex. 2.—Given $\tan A = \dfrac{a}{b}$. Find the other ratios.



Draw the straight lines AC, CB, at right angles to each other, b and a units in length respectively, join AB, then BAC is the

angle required. Then, since $AB = \sqrt{a^2 + b^2}$ we have

$$\sin A = \frac{a}{\sqrt{a^2 + b^2}}, \quad \cos A = \frac{b}{\sqrt{a^2 + b^2}}.$$

Similarly the other ratios may be written. The two results given should be memorized

EXERCISE V.

1. Find the value of the remaining trigonometrical ratios, having given

(1) $\cos A = \frac{12}{13}$. (2) $\tan A = \frac{7}{24}$. (3) $\sec A = 3$.

(4) $\cot A = \dfrac{2mn}{m^2 - n^2}$. (5) $\tan A = 2 \pm \sqrt{3}$. (6) $\tan A = 3 \cot A$.

2. The sine of an obtuse angle is $\frac{3}{5}$. Find the cosine, tangent, and cosecant.

3. The tangent of an angle of a triangle is $-\frac{99}{20}$. Find its sine and cosine. In this example, what is the use of the phrase "of a triangle"?

4. In Ex. 2, Art. 52, how can we determine which sign to place before the radical in the values of the sine and cosine? Must the same sign be used in both cases?

5. The sine of an angle of $70°$ is $\dfrac{a}{c}$. Find the cosine and tangent of $20°$.

6. Given $\sin A = m \cos A$. Find $\sec A$ and $\operatorname{cosec} A$.

7. If $\tan \theta = x$, find the value of $\sin \theta$ and $\cos \theta$, and prove $\sin^2 \theta + \cos^2 \theta = 1$.

8. The perpendicular from the angle A, on the opposite side BC of a triangle is 3 feet, $\sin B = \frac{2}{3}$, and $\sin C = \frac{4}{5}$. Find the sides AB and AC. Show that two triangles can be drawn fulfilling these conditions.

53. Between three quantities only two independent ratios

exist. Four other ratios may be written but their values will clearly depend upon the values of the first two. This is the case with the six trigonometrical ratios of an angle which are formed from only three straight lines; the last four may be expressed in terms of the first two, the sine and the cosine. Thus, denoting the angle XOP (Art. 43), by A, the student can easily prove the following relations:

$$(1)\ \tan A = \frac{\sin A}{\cos A}, \qquad (2)\ \operatorname{cosec} A = \frac{1}{\sin A},$$

$$(3)\ \sec A = \frac{1}{\cos A}, \qquad (4)\ \cot A = \frac{\cos A}{\sin A}.$$

Again, the sides of the triangle OMP are connected by the relation

$$x^2 + y^2 = r^2 \qquad \text{Euc. I., 47.}$$

which furnishes three more important formulæ.

Dividing in succession by r^2, x^2, y^2, we get

$$(5)\ \frac{x^2}{r^2} + \frac{y^2}{r^2} = 1, \quad \text{or,} \quad \cos^2 A + \sin^2 A = 1$$

$$(6)\ 1 + \frac{y^2}{x^2} = \frac{r^2}{x^2}, \quad \text{or,} \quad 1 + \tan^2 A = \sec^2 A$$

$$(7)\ \frac{x^2}{y^2} + 1 = \frac{r^2}{y^2}, \quad \text{or,} \quad 1 + \cot^2 A = \operatorname{cosec}^2 A.$$

These three formulæ, being but different forms of the same equation, give but one independent relation among the ratios. We have, therefore, in all five independent relations between six quantities, and consequently when the value of any one is given, corresponding values of the others may be found. But one of these relations is of the *second degree*, and consequently when this one is involved there will be *two* solutions. The full meaning of this double result cannot be given at once, but will become evident as the student proceeds with the subject. The seven formulæ should be carefully memorized.

54. The equations of Art. 53 enable us to express all the ratios in terms of any one of them, the process being purely algebraical. **As** an example, we will express them all in terms of the sine.

From (5) $\cos A = \sqrt{1 - \sin^2 A}$

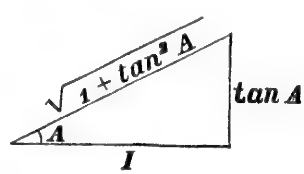
from (1) and (5) $\tan A = \dfrac{\sin A}{\cos A} = \dfrac{\sin A}{\sqrt{1 - \sin^2 A}}$

from (3) $\sec A = \dfrac{1}{\cos A} = \dfrac{1}{\sqrt{1 - \sin^2 A}},$

from (4) and (5) $\cot A = \dfrac{\cos A}{\sin A} = \dfrac{\sqrt{1 - \sin^2 A}}{\sin A}$

and from (2) $\operatorname{cosec} A = \dfrac{1}{\sin A}.$

Again, we will express them all in terms of the tangent by the



geometrical method. From the given diagram the ratios can be written by inspection; thus:

$$\sin A = \frac{\tan A}{\sqrt{1 + \tan^2 A}}, \quad \cos A = \frac{1}{\sqrt{1 + \tan^2 A}}, \quad \sec A = \sqrt{1 + \tan^2 A}$$

$$\operatorname{cosec} A = \frac{\sqrt{1 + \tan^2 A}}{\tan A}, \quad \cot A = \frac{1}{\tan A}.$$

In employing the geometrical method, take as the unit of length the line which represents the denominator of the given ratio.

The results are most easily obtained by the geometrical method, but certainty of their universal truth can be obtained only by examining an angle in each quadrant in succession.

But the algebraical process, being conducted without reference to any particular figure, gives results known at once to be perfectly general, since the original equations are universally true.

55. The formulæ of Art. 53 are extensively used to establish identities, solve equations, etc. We give a few examples.

Ex. 1. Simplify $(\sec A + \operatorname{cosec} A)^2 - (\tan A + \cot A)^2$.

$$\begin{aligned}
&(\sec A + \operatorname{cosec} A)^2 - (\tan A + \cot A)^2 \\
&\quad = \sec^2 A + 2 \sec A \operatorname{cosec} A + \operatorname{cosec}^2 A - \tan^2 A - 2 - \cot^2 A \\
&\quad = (\sec^2 A - \tan^2 A) + (\operatorname{cosec}^2 A - \cot^2 A) + 2 \sec A \operatorname{cosec} A - 2 \\
&\quad = 1 + 1 + 2 \sec A \operatorname{cosec} A - 2 \\
&\quad = 2 \sec A \operatorname{cosec} A.
\end{aligned}$$

Ex. 2. Prove $(\sec \theta - \operatorname{cosec} \theta)(1 + \cot \theta + \tan \theta) = \sin \theta \sec^2 \theta - \cos \theta \operatorname{cosec}^2 \theta$.

$$\begin{aligned}
&(\sec \theta - \operatorname{cosec} \theta)(1 + \cot \theta + \tan \theta) \\
&\quad = \left(\frac{1}{\cos \theta} - \frac{1}{\sin \theta}\right)\left(1 + \frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta}\right) \\
&\quad = \frac{\sin \theta - \cos \theta}{\cos \theta \sin \theta} \cdot \frac{\sin \theta \cos \theta + \cos^2 \theta + \sin^2 \theta}{\sin \theta \cos \theta} \\
&\quad = \frac{\sin^3 \theta - \cos^3 \theta}{\sin^2 \theta \cos^2 \theta} = \frac{\sin \theta}{\cos^2 \theta} - \frac{\cos \theta}{\sin^2 \theta} \\
&\quad = \sin \theta \sec^2 \theta - \cos \theta \operatorname{cosec}^2 \theta.
\end{aligned}$$

Ex. 3. Given $\cot \theta + \operatorname{cosec} \theta = 5$. Find $\cos \theta$.

We have $\dfrac{\cos \theta}{\sin \theta} + \dfrac{1}{\sin \theta} = 5$

therefore $\cos \theta + 1 = 5 \sin \theta$

squaring, $\cos^2 \theta + 2 \cos \theta + 1 = 25 \sin^2 \theta$

$= 25 - 25 \cos^2 \theta$

from which $26 \cos^2 \theta + 2 \cos \theta - 24 = 0$

factoring, $(13 \cos \theta - 12)(\cos \theta + 1) = 0$

therefore $\cos \theta = \frac{12}{13}$, or -1.

It will be readily perceived that the above is an ordinary

vers $A = 1 - \cos A$,
covers $A = 1 - \sin A$

quadratic equation, the unknown quantity being $\cos\theta$. In complicated examples the work is facilitated by writing a single letter x for the given ratio.

EXERCISE VI.

Prove the following identities:

1. $\sin A . \cot A . \sec A = 1$.
2. $\cos A . \tan A . \operatorname{cosec} A = 1$.
3. $\operatorname{vers} A\,(1+\cos A) = \sin^2 A$.
4. $\operatorname{covers} A\,(1+\sin A) = \cos^2 A$.
5. $\sin^2 A + \operatorname{vers}^2 A = 2(1-\cos A)$.
6. $\cos^2 A + \operatorname{covers}^2 A = 2(1-\sin A)$.
7. $(\tan A + \cot A)\sin A \cos A = 1$.
8. $(\tan A + \cot A)^2 = \sec^2 A + \operatorname{cosec}^2 A$.
9. $\sin^2 A - \cos^2 B = \sin^2 B - \cos^2 A$.
10. $\tan^2 A - \tan^2 B = \sec^2 A - \sec^2 B$.
11. $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$.
12. $\tan^4\theta + \sec^4\theta = 1 + 2\tan^2\theta\sec^2\theta$.
13. $\tan^4\theta + \tan^2\theta = \sec^4\theta - \sec^2\theta$.
14. $\cot^4\theta + \cot^2\theta = \operatorname{cosec}^4\theta - \operatorname{cosec}^2\theta$.
15. $\sin\theta\tan\theta + \cos\theta\cot\theta = \sec\theta\operatorname{cosec}\theta\,(1-\sin\theta\cos\theta)$.
16. $\sin^2\theta\tan\theta + \cos^2\theta\cot\theta + 2\sin\theta\cos\theta = \tan\theta + \cot\theta$
$= \sec\theta\operatorname{cosec}\theta$.
17. $\sin^6 x + \cos^6 x = 1 - 3\sin^2 x\cos^2 x$.
18. $\sin^6 x - \cos^6 x = (1-2\cos^2 x)(\sin^2 x + \cos^4 x)$
$= (2\sin^2 x - 1)(1 - \sin^2 x\cos^2 x)$.
19. $(\sin A\cos B + \cos A\sin B)^2 + (\cos A\cos B - \sin A\sin B)^2 = 1$.
20. $(1+\sin A+\cos A)^2 = 2(1+\sin A)(1+\cos A)$.
21. $(1+\sin A-\cos A)^2 + (1+\cos A-\sin A)^2 = 4(1-\sin A\cos A)$.
22. $\sin\theta = \sqrt{1-\cos^2\theta} = \dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \dfrac{1}{\sqrt{1+\cot^2\theta}}$

$$= \frac{\sqrt{\sec^2\theta - 1}}{\sec\theta} = \frac{1}{\operatorname{cosec}\theta}.$$

23. Write a set of equations similar to the preceding, beginning with each ratio in succession.

24. $\cos A(\tan A+2)(2\tan A+1)=2\sec A+5\sin A$.

25. $\sin A(\cot A+3)(3\cot A+1)=3\operatorname{cosec} A+10\cos A$.

26. $\frac{1}{3}(\cos^6 A+\sin^6 A)-\frac{1}{4}(\cos^2 A-\sin^2 A)^2=\frac{1}{12}$.

27. $\sin A(1+\tan A)+\cos A(1+\cot A)=\sec A+\operatorname{cosec} A$.

28. If $\cos A=\cot B$, then $\sin^2 A\sin^2 B=\sin^2 B-\cos^2 B$.

29. $\sqrt{\dfrac{1-\sin A}{1+\sin A}}=\sec A-\tan A$, and $\sqrt{\dfrac{1-\cos A}{1+\cos A}}=\dfrac{\sin A}{1+\cos A}$.

30. $\dfrac{1+\cos\theta}{1-\cos\theta}-\dfrac{1-\cos\theta}{1+\cos\theta}=4\cot\theta\operatorname{cosec}\theta$.

31. $\dfrac{\sec\theta+\tan\theta}{\sec\theta-\tan\theta}-\dfrac{\sec\theta-\tan\theta}{\sec\theta+\tan\theta}=\dfrac{1+\sin\theta}{1-\sin\theta}-\dfrac{1-\sin\theta}{1+\sin\theta}$.

32. Given $\tan\theta+\cot\theta=4$; find $\tan\theta$ and $\sec\theta$.

33. Given $3\tan^2\theta-4\sin^2\theta=1$; find $\sin\theta$ and $\cot\theta$.

34. Given $3\sin^2\theta-\cos^2\theta=6\cot^2\theta$; find $\cos\theta$ and $\tan\theta$.

35. Given $3\sin^2\theta-\cos^2\theta+(\sqrt{5}+1)\sin\theta=\frac{1}{2}(3-\sqrt{5})$; find $\sin\theta$ and $\cot\theta$.

36. If $12\sec^2\theta=6+17\tan\theta$, find the value of $\tan\theta+\cot\theta$.

37. If $\tan\theta=\dfrac{a}{b}$, find the value of $a\sin\theta+b\cos\theta$.

38. If $\tan x+ab\cot x=a+b$, find $\tan x$.

39. If $\sin x\tan y=\tan a$, $\cot x\cos y=\cot b$, find $\tan x$ and $\tan y$.

Eliminate θ from the following sets of equations:

40. $x=a\cos\theta$, $y=b\sin\theta$.

41. $x=a\sec\theta$, $y=b\tan\theta$.

42. $x=\sin\theta$, $y=\tan\theta$.

43. $x=a\cos^3\theta$, $y=a\sin^3\theta$.

44. $x=\cos\theta\cot\theta$, $y=\sin\theta\tan\theta$.

45. $x=\sec\theta\tan\theta$, $y=\operatorname{cosec}\theta\cot\theta$.

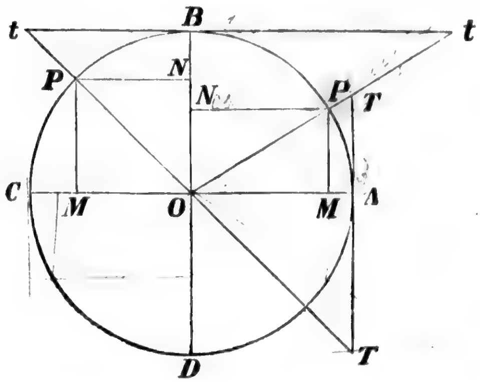
46. $x\cos\theta-y\sin\theta=a$, $x\sin\theta+y\cos\theta=b$.

47. $\sin\theta+\cos\theta=a$, $\tan\theta+\sec\theta=b$.

56. It will be instructive to compare the definitions of the trigonometrical ratios with the meanings attached to the terms sine, cosine, etc., by the early writers on trigonometry.

Let $ABCD$ be a circle, AOC, BOD, two diameters at right angles to each other, AP any arc of this circle. From P, either extremity of the arc, draw PM perpendicular to the radius OA at the other extremity; draw also AT, Bt, touching the circle at A, B, and meeting the radius OP produced in T, t; then

MP = sine of arc AP.	OM = cosine of arc AP.
AT = tangent of arc AP.	Bt = cotangent of arc AP.
OT = secant of arc AP.	Ot = cosecant of arc AP.
MA = versed sine of arc AP.	BN = coversed sine of arc AP.



Thus, the old system dealt with *arcs* and *lines*, which we have replaced by *angles* and *ratios*, and it will be observed that the versed sine and coversed sine belong properly to this method.

If the radius of the circle be unity, then the length of the arc AP becomes the circular measure of the angle AOP, and the lengths of the various lines become its trigonometrical ratios. If r be the radius, θ the angle, then

$$MP = r \sin \theta, \quad AP = r\,\theta, \quad AT = r \tan \theta.$$

57. The preceding diagram contains the proper construction when the angle (or arc) lies in either the first or the second quadrant. The student should draw the corresponding figures for the third and fourth quadrants. Then, calling the radius of the circle r, and carefully distinguishing between positive and negative directions, as the point P moves around the circle we easily see the truth of the following:

1. The values of both sine and cosine lie between $+r$ and $-r$ for all values of the circular arc.
2. The secant and cosecant may each have any value which is not between $+r$ and $-r$.
3. The tangent and cotangent may each have any numerical value, either positive or negative.
4. The versed sine and coversed sine are always positive, and may have any value between 0 and $2r$.
5. In the first quadrant all the ratios are positive, and the groups sine, tangent, secant; cosine, cotangent, cosecant, are each in ascending order of magnitude.
6. The sine and cosecant are positive in the first and second quadrants (the semicircle above AOC), negative in the other two.
7. The cosine and secant are positive in the first and fourth quadrants (the semicircle to the right of BOD), negative in the other two.
8. The tangent and cotangent are positive in the first and third quadrants (the quadrants opposite each other), negative in the other two.
9. As the angle increases through the first quadrant the sine, tangent, and secant, also increase; but the cosine, cotangent, and cosecant diminish.

58. The meaning of the prefix "co" found in three of the trigonometrical ratios may now be given. Since the arcs AP, PB, together make up a quadrant, they are said to be complementary in the same way as the angles AOP, POB, which they subtend. If PN be drawn perpendicular to OB, then by definition, NP, which equals OM, is the sine, Bt the tangent, and Ot

the secant of the arc PB, the complement of AP. Thus "co-sine" is an abbreviation of "complement sine," meaning, "sign of the complement." Similarly for the other ratios.

59. Since the cosine, cotangent, and cosecant of one angle are the sine, tangent, and secant of another (its complement), general properties of the former group, individually or collectively, will be similarly true of the latter. Good examples are found in Art. 57, or in almost any part of the subject.

EXERCISE VII.

1. Using the *linear* values of the sine, cosine, etc., prove the following, in which θ denotes any circular arc, r the radius of the circle:

(1) $\sin^2\theta + \cos^2\theta = r^2$. (2) $r^2 + \tan^2\theta = \sec^2\theta$.
(3) $r^2 + \cot^2\theta = \operatorname{cosec}^2\theta$. (4) $r\sin\theta = \cos\theta \,.\, \tan\theta$.

2. In the fig of Art. 56, prove the following:

(1) $OM \,.\, OT = OP^2$. (2) $AT \,.\, Bt = OP^2$.
(3) $OM \,.\, AT = OP \,.\, PM$. (4) $PM \,.\, Bt = OP \,.\, OM$.

State in each case the corresponding equations in terms of the ratios of the angle AOP.

3. Prove $PM \,.\, Bt \,.\, OT = OP^3$. State the corresponding theorem with regard to the angle BOP. Express each in terms of trigonometrical ratios.

4. Find the length of a circular arc whose sine and cosine are each 5 feet in length.

(1) sine and cosine each positive
(2) sine positive, cosine negative.
(3) sine and cosine each negative.
(4) sine negative, cosine positive.

5. The tangent of a circular arc, radius 5 feet, is 12 feet in length; find the sine, cosine, and cotangent. Find two arcs which satisfy the conditions and the difference in their lengths.

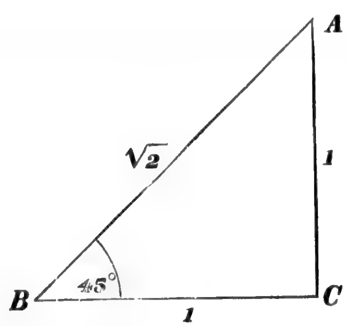
CHAPTER IV.

RATIOS OF PARTICULAR ANGLES.

60. With a few exceptions the trigonometrical ratios are incommensurable quantities, and the calculation of their approximate numerical values is a work of some difficulty and much labor, but their values for a few angles may be found from simple geometrical constructions. We shall frequently use their values in the surd form, in which they are most easily obtained, though in practical work decimals alone are employed.

61. In the study of trigonometry there are many facts and formulæ which must be committed to memory. This is most easily done by associating them with a geometrical figure. For example, the values of the ratios given in Arts. 62-65 can be immediately written whenever wanted by simply remembering the lengths of the sides of the triangles from which they were obtained. Also the angles being positive and acute, no distinction of direction is necessary.

62. *To find the values of the trigonometrical ratios for an angle of 45°.*

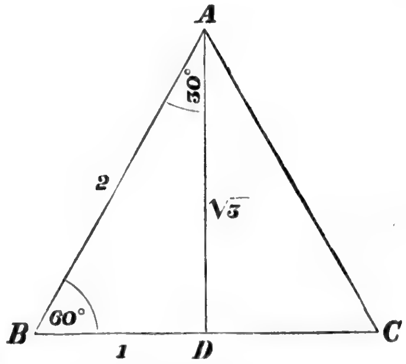


Construct a right-angled isosceles triangle ACB, of which C is the right angle, then each of the angles A and B is half a right angle. Euc. I. 5, 32.

Take $AC = CB$ as the unit of length, then $AB = \sqrt{2}$. Euc. I. 47.

Therefore
$$\sin 45^\circ = \frac{AC}{AB} = \frac{1}{\sqrt{2}} \qquad \cos 45^\circ = \frac{BC}{AB} = \frac{1}{\sqrt{2}}$$
$$\tan 45^\circ = \frac{AC}{CB} = 1 \qquad \cot 45^\circ = \frac{BC}{AC} = 1$$
$$\sec 45^\circ = \frac{AB}{BC} = \sqrt{2} \qquad \operatorname{cosec} 45^\circ = \frac{AB}{AC} = \sqrt{2}$$

63. *To find the ratios for angles of 30° and 60°.*



Construct an equilateral triangle ABC, bisect the angle A by the line AD, then the angles A, B, C, being all equal, each contains 60°, and therefore BAD contains 30°. Euc. I. 32.

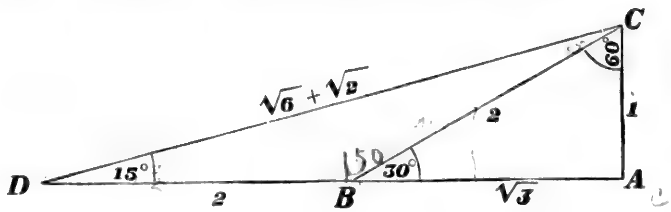
Take BD as the unit of length, then $BA = 2$, and $AD = \sqrt{3}$.

Therefore
$$\sin 60^\circ = \cos 30^\circ = \frac{AD}{AB} = \frac{\sqrt{3}}{2}; \qquad \cos 60^\circ = \sin 30^\circ = \frac{BD}{AB} = \frac{1}{2}$$
$$\tan 60^\circ = \cot 30^\circ = \frac{AD}{DB} = \sqrt{3}; \qquad \cot 60^\circ = \tan 30^\circ = \frac{BD}{AD} = \frac{1}{\sqrt{3}}$$
$$\sec 60^\circ = \operatorname{cosec} 30^\circ = \frac{AB}{BD} = 2; \qquad \operatorname{cosec} 60^\circ = \sec 30^\circ = \frac{AB}{AD} = \frac{2}{\sqrt{3}}$$

64. *To find the ratios for angles of 15° and 75°.*

Let ABC be the half of an equilateral triangle so that $\angle ACB = 60°$, $\angle ABC = 30°$, $\angle BAC = 90°$. Produce AB to D, making $BD = BC$; join DC. Then, the angles BDC, BCD being equal, each of them is half the angle ABC, *i.e.* 15°, and $\angle DCA$ is 75°. Euc. I. 32.

Take AC as the unit of length, then $BD = BC = 2$, and $AB = \sqrt{3}$.



Also $CD^2 = DA^2 + AC^2 = (2+\sqrt{3})^2 + 1^2 = 8 + 4\sqrt{3} = (\sqrt{6}+\sqrt{2})^2$
Therefore $CD = \sqrt{6} + \sqrt{2}$.

Then $\sin 15° = \cos 75° = \dfrac{AC}{CD} = \dfrac{1}{\sqrt{6}+\sqrt{2}} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

$\cos 15° = \sin 75° = \dfrac{AD}{CD} = \dfrac{2+\sqrt{3}}{\sqrt{6}+\sqrt{2}} = \dfrac{\sqrt{6}+\sqrt{2}}{4}$

$\tan 15° = \cot 75° = \dfrac{AC}{AD} = \dfrac{1}{2+\sqrt{3}} = 2 - \sqrt{3}$

$\cot 15° = \tan 75° = \dfrac{AD}{AC} = 2 + \sqrt{3}$

$\sec 15° = \operatorname{cosec} 75° = \dfrac{CD}{AD} = \dfrac{\sqrt{6}+\sqrt{2}}{2+\sqrt{3}} = \sqrt{6} - \sqrt{2}$

$\operatorname{cosec} 15° = \sec 75° = \dfrac{CD}{AC} = \sqrt{6} + \sqrt{2}$

When the sine of any angle is known, a construction similar to the preceding will give the ratios for half that angle. This principle is more fully considered in Art. 73.

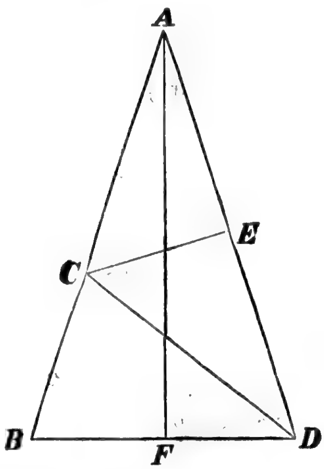
65. *To find the ratios for angles of 18°, 36°, 54° and 72°.*

Let ABD be an isosceles triangle having each of the angles at B and D double the angle A.

Then $\angle BAD = \frac{1}{5}$ of 2 right angles $= 36°$. Bisect BAD by AF, which will also bisect BD at right angles.

Then $\angle BAF = 18°$, and angle $ABF = 72°$.

Bisect $\angle BDA$ by DC, meeting AB in C; denote AB by a, and $BD = CD = CA$ by x.



From similar triangles ABD, DBC,

$$AB : BD :: BD : BC, \text{ or } a : x :: x : a - x. \qquad \text{Euc. VI. 4.}$$

Therefore $$a^2 - ax = x^2$$

or $$\frac{x^2}{a^2} + \frac{x}{a} - 1 = 0$$

from which $$\frac{x}{a} = \frac{-1 \pm \sqrt{5}}{2}.$$

Then $$\sin 18° = \cos 72° = \frac{BF}{AB} = \frac{1}{2} \cdot \frac{x}{a} = \frac{\sqrt{5} - 1}{4}$$

and $$\cos 18° = \sin 72° = \frac{AF}{AB} = \frac{1}{a}\sqrt{a^2 - \frac{x^2}{4}} = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}$$

The positive sign only is to be taken in the value of $\dfrac{x}{a}$, since x and a are both positive.

Again, draw CE perpendicular to AD,

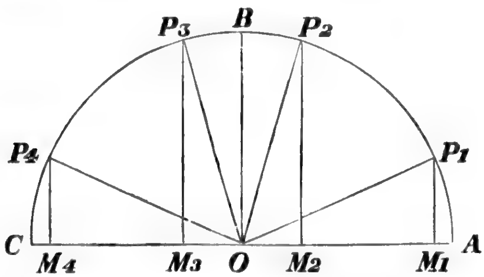
Then AD is bisected, $\angle CDE = 36°$, and $\angle DCE = 54°$

$$\text{Therefore } \sin 54° = \cos 36° = \frac{DE}{CD} = \frac{1}{2} \cdot \frac{a}{x} = \frac{1}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{4}$$

$$\text{and} \qquad \cos 54° = \sin 36° = \frac{CE}{CD} = \frac{1}{x}\sqrt{x^2 - \frac{a^2}{4}} = \frac{1}{4}\sqrt{10 - 2\sqrt{5}}.$$

The values of the remaining ratios may be deduced from those of the sine and cosine here given.

66. *To find the ratios of 0°, 90° and 180°.*



Let a straight line OP revolve in a semicircle ABC, and let P_1, P_2, P_3, P_4 be different positions of P, from which perpendiculars are drawn to the diameter AOC as in the figure. From the triangles thus found the ratios of the corresponding angles may be written, excepting at the points A, B, C, where the triangle becomes a straight line. But the general definition, Art. 43, derived from the co-ordinates of the point P, still holds. If r be the radius, the co-ordinates of A are r, 0; of B, 0, r; of C, $-r$, 0. Then remembering that the revolving line is always positive, we get the following results :

ANGLE	0°	90°	180°
sine	$\frac{0}{r}=0.$	$\frac{r}{r}=1.$	$\frac{0}{r}=0.$
cosine	$\frac{r}{r}=1.$	$\frac{0}{r}=0.$	$\frac{-r}{r}=-1.$
tangent	$\frac{0}{r}=0.$	$\frac{r}{0}=\infty.$	$\frac{0}{-r}=0.$
cosecant	$\frac{r}{0}=\infty.$	$\frac{r}{r}=1.$	$\frac{r}{0}=\infty.$
secant	$\frac{r}{r}=1.$	$\frac{r}{0}=\infty.$	$\frac{r}{-r}=-1.$
cotangent	$\frac{r}{0}=\infty.$	$\frac{0}{r}=0.$	$\frac{-r}{0}=-\infty.$

67. The preceding article gives excellent illustrations of certain algebraical operations. Thus, when we say $\tan 90^\circ=\frac{r}{0}=\infty$, we mean that as the point P_2 approaches B, the quotient of the length of the perpendicular $P_2 M_2$ by the length of the base OM_2 becomes greater and greater, and that before P_2 reaches B this quotient becomes greater than any finite quantity. Again, when P_2 is indefinitely near to B the tangent is indefinitely great and *positive*, but if P_3 be brought indefinitely near to B the tangent is indefinitely great and *negative*. The tangent thus passes instantly from $+\infty$ to $-\infty$, as the point P passes through B, just as the cosine passes from $+0$ to -0. When a variable quantity *changes sign* it must first become either *zero* or *infinite;* the cosine is an example of the former, the tangent, of the latter. The reader should compare Art. 57, and observe how the *linear* tangent becomes infinite in *length* and changes *direction*, whilst the *ratio* tangent becomes infinite *numerically* and changes *sign* as the point P passes from the first quadrant to the second.

68. It is important that the student should be able to state accurately in words, the changes which each of the ratios undergoes as the revolving line passes through each quadrant in

succession. We give such a statement for the second quadrant as an example.

As the angle increases from 90° to 180°:

1. The sine is positive, and decreases from 1 to 0.

2. The cosine is negative, and increases numerically from 0 to -1.

3. The tangent is negative, and decreases numerically from $-\infty$ to 0.

4. The cosecant is positive, and increases from 1 to ∞.

5. The secant is negative, and decreases numerically from $-\infty$ to 1.

6. The cotangent is negative, and increases numerically from 0 to $-\infty$.

A similar statement should be given for each of the quadrants.

69. A trigonometrical equation is a statement of equality between two expressions involving the ratios of an unknown angle, and to solve the equation is to find the angle whose ratios satisfy the given relation. The solution generally requires three distinct steps:

1. Express the different ratios which occur in terms of a single ratio.

2. Consider this ratio the unknown quantity, and solve the resulting equation by the ordinary rules of algebra.

3. Write down the smallest positive angle whose ratio is known to have the value thus found.

We give a few simple examples.

Ex. 1.—Given $\tan\theta + \cot\theta = 2$, to find θ.

Expressing all in terms of the tangent, we have

$$\tan\theta + \frac{1}{\tan\theta} = 2.$$

Simplifying, $\tan^2\theta - 2\tan\theta + 1 = 0$

extracting sq. root, $\tan\theta - 1 = 0$

or $\tan\theta = 1$

therefore $\theta = 45^\circ$. Art. 62.

Ex. 2.—Solve $\tan^2\theta + \sec\theta = 5$.

Since $1 + \tan^2\theta = \sec^2\theta$, $\therefore\ \tan^2\theta = \sec^2\theta - 1$

and we have $\qquad \sec^2\theta - 1 + \sec\theta = 5$

or $\qquad \sec^2\theta + \sec\theta - 6 = 0$

factoring, $\qquad (\sec\theta + 3)(\sec\theta - 2) = 0$

therefore $\qquad \sec\theta = 2$, or -3

and $\qquad \theta = 60^\circ$. $\qquad$ Art. 63.

Thus far we have not met with an angle whose secant is -3. The student can easily draw the angle geometrically, and show that it lies between 90° and 180°, but we have not as yet the means to obtain its exact magnitude.

Ex. 3.—Given $2\sin A = \sqrt{3}\tan A$, to find A.

Writing both terms on the same side of the equation and factoring, we get

$$\sin A\left(2 - \frac{\sqrt{3}}{\cos A}\right) = 0$$

or $\qquad \sin A\,(2\cos A - \sqrt{3}) = 0$

therefore $\qquad \sin A = 0$, or $\cos A = \dfrac{\sqrt{3}}{2}$.

from which $\qquad A = 0$, or $\quad A = 30^\circ$. $\qquad$ Arts. 66, 63.

Ex. 4.—Given $\sin\theta + \operatorname{cosec}\theta = 2\frac{1}{2}$, to find θ.

Expressing all in terms of $\sin\theta$ and simplifying we get

$$2\sin^2\theta - 5\sin\theta + 2 = 0.$$

Factoring, $\qquad (2\sin\theta - 1)(\sin\theta - 2) = 0$

from which $\qquad \sin\theta = \frac{1}{2}$

therefore $\qquad \theta = 30^\circ$.

The equation is also satisfied *algebraically* by $\sin\theta = 2$, but this is impossible *trigonometrically*, since the sine of an angle is never greater than unity.

EXERCISE VIII.

1. From the results of Art. 62, write the ratios of $135°$ and $-45°$.

2. By the method of Art. 63 obtain the ratios of $120°$ and $150°$. Also those of $210°$ and $-30°$.

3. Apply Art. 64 to obtain the ratios of $22\frac{1}{2}°$ and $67\frac{1}{2}°$.

4. Write down all the ratios of $270°$.

5. Find all the values of θ between 0 and $360°$ for which $\cos^2\theta = \frac{1}{2}$.

6. Find all the values of $\cos\left(2n\pi \pm \frac{\pi}{6}\right)$ as n is given different integral values, positive or negative.

7. Find all the values of $\tan\left(n\pi + \frac{\pi}{4}\right)$ where n is any integer.

8. If $A = 45°$, $B = 30°$, verify the formulæ

$$\sin(A+B) = \sin A\cos B + \cos A\sin B,$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B.$$

9. Prove the following identities:

(1) $\sin 30° + \sin 60° = \sqrt{2}.\cos 15°$. (2) $\cos\frac{\pi}{6} - \sin\frac{\pi}{6} = \sqrt{2}.\sin\frac{\pi}{12}$.

(3) $(\sin 60° - \sin 45°)(\cos 45° + \cos 30°) = \sin^2 30°$.

(4) $\cos 36° - \sin 18° = \frac{1}{2}$. (5) $4\sin 18°\cos 36° = 1$.

(6) $\sin 45° = 3\sin 15° - 4\sin^3 15°$. (7) $\sin 36°\cos 18° = \sin 54° - \frac{1}{4}$.

(8) $3\tan 15° + 3\tan^2 15° - \tan^3 15° = 1$.

10. From the result of Art. 65 prove the following:

(1) $\tan 18° = \cot 72° = \sqrt{1 - \frac{2}{5}\sqrt{5}}$.

(2) $\cot 18° = \tan 72° = \sqrt{5 + 2\sqrt{5}}$

(3) $\sec 18° = \operatorname{cosec} 72° = \sqrt{2 - \frac{2}{5}\sqrt{5}}$.

(4) $\operatorname{cosec} 18° = \sec 72° = \sqrt{5} + 1$.

(5) $\tan 36^\circ = \cot 54^\circ = \sqrt{5 - 2\sqrt{5}}$.

(6) $\cot 36^\circ = \tan 54^\circ = \sqrt{1 + \frac{2}{5}\sqrt{5}}$.

(7) $\sec 36^\circ = \operatorname{cosec} 54^\circ = \sqrt{5} - 1$.

(8) $\operatorname{cosec} 36^\circ = \sec 54^\circ = \sqrt{2 + \frac{2}{5}\sqrt{5}}$.

11. Solve the following equations:

(1) $2 \sin \theta = \tan \theta$. (2) $3 \sin^2 \theta = \cos^2 \theta$.

(3) $\tan \theta + \cot \theta = \frac{4}{\sqrt{3}}$. (4) $\cos \theta = 3 \cot \theta$.

(5) $\cot \theta = 2 \cos \theta$. (6) $\operatorname{cosec} \theta - 2 = 4 \sin \theta$.

(7) $\sec \theta \tan \theta = 2\sqrt{3}$. (8) $3 \sec^4 \theta - 10 \tan^2 \theta = 2$.

(9) $\sin^3 \theta + \cos^3 \theta = 0$. (10) $2 \tan \theta + \sec^2 \theta = 2$.

(11) $\sin \theta + \cos \theta = \sqrt{\frac{3}{2}}$. (12) $\cos^2 \theta - \sin^2 \theta = \frac{\sqrt{5}+1}{4}$.

(13) $2 \sin \theta \tan \theta + 1 = \tan \theta + 2 \sin \theta$.

(14) $3 \cos^2 \theta - \sin^2 \theta + (\sqrt{3} + 1)(1 - 2 \cos \theta) = 0$.

(15) $\sin (A - B) = \frac{1}{2}$ and $\cos (A + B) = \frac{1}{2}$.

(16) $\tan (A + B) = \sqrt{3}$, and $\tan (A - B) = 1$.

(17) $\tan (A + B) = 2 - \sqrt{3}$, and $\cos (A - B) = \frac{\sqrt{3} - 1}{2\sqrt{2}}$.

(18) $\sin (A + B + C) = 1$, $\cos (A + B - C) = 1$, $\tan (A - B) = 2 - \sqrt{3}$.

12. Find $\tan 7\frac{1}{2}^\circ$, and $\tan 37\frac{1}{2}^\circ$.

13. If the sine of an angle be greater than $\frac{1}{\sqrt{2}}$, and its cosine greater than $\frac{1}{2}$, between what limits does the angle lie?

14. Trace the changes in the value of the following as θ changes from 0 to $\frac{\pi}{2}$ and from $\frac{\pi}{2}$ to π:

(1) $1 - \cos \theta$. (2) $\cos \theta - \sin \theta$. (3) $\tan \theta + \cot \theta$.

15. As the trigonometrical ratios change from positive to negative, or from negative to positive, state which of them can pass through the value (1) zero only, (2) infinity only, (3) either zero or infinity.

16. Determine whether the equations

$$(1)\ \sec\theta = \frac{4ab}{(a+b)^2}, \qquad (2)\ \tan^2\theta = \frac{4ab}{(a+b)^2},$$

are possible when a and b denote unequal numbers of the same sign. Also when they denote equal numbers of opposite sign.

17. Show geometrically that $\sin 2A < 2\sin A$, and that $\sin(A+B) < \sin A + \sin B$.

18. Eliminate θ from the equations:

$$(1)\ \begin{array}{l}\operatorname{cosec}\theta - \sin\theta = a.\\ \sec\theta - \cos\theta = b.\end{array} \qquad (2)\ \begin{array}{l}a\cos^2\theta + b\sin^2\theta = m.\\ a\sin^2\theta - b\cos^2\theta = n.\end{array}$$

19. Eliminate θ and ϕ from the equations:

$$a\sin^2\theta + b\cos^2\theta = m. \qquad b\sin^2\phi + a\cos^2\phi = n.$$

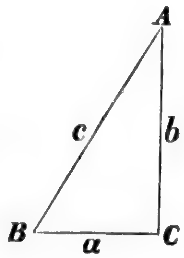
$$a\tan\theta = b\tan\phi.$$

Practical Applications.

70. Thus far we have been engaged in determining the trigonometrical ratios of various angles from the known sides of right-angled triangles. We shall now reverse the process, and show that by the aid of those results when one side and one angle are known, the remaining sides and angle may easily be found.

71. The angles of a triangle are usually denoted by the capital letters A, B, C, and the sides opposite them by the small letters a, b, c, and in the case of right-angled triangles, C denotes the right angle. This notation will be followed unless otherwise specified.

72. *Given one side and one angle of a right-angled triangle to find the other sides.*



From the right-angled triangle ABC, $\frac{b}{c} = \sin B$

$\therefore b = c\sin B$, so that when c and $\sin B$ are each known the value of b can be immediately

found. Similarly, each of the following results may be easily verified.

$$\begin{array}{lll} b = c \sin B & a = c \sin A & c = b \operatorname{cosec} B \\ \quad = c \cos A & \quad = c \cos B & \quad = b \sec A \\ \quad = a \tan B & \quad = b \tan A & \quad = a \operatorname{cosec} A \\ \quad = a \cot A. & \quad = b \cot B. & \quad = a \sec B. \end{array}$$

The student should not attempt to memorize the above equations literally, but endeavor to acquire facility in writing them from the figure.

Ex. 1.—Given $c = 25$, $B = 15°$, find the other parts.

Since $A + B = 90°$, $\therefore A = 90° - 15° = 75°$.

To find a and b, we have

$$a = c \cos B, \qquad b = c \sin B \qquad \text{Art. 72.}$$

$$= \frac{25(\sqrt{6}+\sqrt{2})}{4} = 24.1481. \qquad = \frac{25(\sqrt{6}-\sqrt{2})}{4} = 6.4705.$$

Ex. 2.—Given $a = 10$, $A = 22\frac{1}{2}°$, find the other sides.

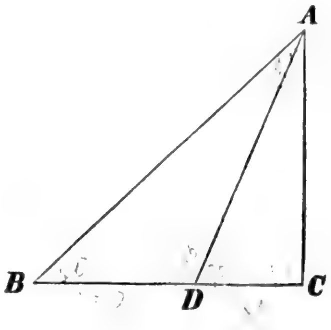
$$b = a \cot A \qquad c = a \operatorname{cosec} A \qquad \text{Art. 72.}$$

$$= 10(\sqrt{2}+1) = 2.4142. \qquad = 10\sqrt{4+2\sqrt{2}} = 26.1312.$$

Ex. 3.—Given $BD = 100$, $B = 60°$, $D = 75°$, to find DC and CA.

Denote CA by x, CD by y:

Then $$\frac{y+100}{x} = \cot 60° = \frac{1}{\sqrt{3}} \qquad \frac{y}{x} = \cot 75° = 2 - \sqrt{3}$$



from which $$\frac{100}{x}=\frac{1}{\sqrt{3}}-(2-\sqrt{3})=\frac{4-2\sqrt{3}}{\sqrt{3}}$$

or $$x=\frac{100\sqrt{3}}{4-2\sqrt{3}}=\frac{100\sqrt{3}(4+2\sqrt{3})}{4}=323.205$$

and $$y=x(2-\sqrt{3})=\frac{100\sqrt{3}}{2}=86.602.$$

EXERCISE IX.

1. Given $c=20$, $B=30°$, find a, b, and A.

2. Given $a=15$, $A=15°$, find b and c.

3. Given $a=\sqrt{10}$, $B=18°$, find b and c.

4. Given $b=\frac{1}{\sqrt{2}}$, $B=22\frac{1}{2}°$, find c, a, and the perpendicular from the right angle on the hypothenuse.

5. Find the length of the shadow of a vertical rod 4 feet long, when the sun is 30° degrees above the horizon.

6. Find the height of a perpendicular flag-pole which subtends an angle of $67\frac{1}{2}°$ at a distance of 50 feet.

7. What length of rope will be necessary to extend from the top of a pole 50 feet high to the ground, the rope making an angle of 75° to the horizon?

8. The vertical angle of an isosceles triangle is 108°, and the base is 100. Find the side and the area.

9. Find the area of a right-angled triangle, one acute angle being 30° and the perpendicular from the right angle on the hypothenuse being p.

10. Find the length of the shadow of a pole 20 feet long when the sun is 30° above the horizon, the pole being inclined 45° from the vertical, (1) towards the sun, (2) from the sun.

11. The base angles of a triangle are 75° and 45°, and the perpendicular from the vertex on the base is 3 feet. Find the two sides of the triangle and its area.

12. A man 6 feet high standing in front of a lamp post $7\frac{1}{2}$ feet high observes that his shadow is 20 feet long. Find how far he is from the post, and by how much his shadow will be shortened by his moving 9 inches nearer the post.

13. A sphere 12 inches in diameter lies on a level plane. How far from the point in which it touches the plane will be the extremity of its shadow when the sun is $30°$ high?

14. The length of the shadow of a vertical rod is $\sqrt{3}$ times the length of the rod. Show that by turning the rod through a certain angle its shadow may be made double its own length but not greater; also find the required angle.

15. A man walking along a straight road observes a house in a direction making $22\frac{1}{2}°$ with the road, but after walking one mile its direction makes an angle of $30°$ with the road. How far is the house from the road?

16. Two ropes of 40 and 50 feet in length respectively are stretched from the top of a perpendicular pole to the ground; the former making an angle of $54°$ with the horizon. Find the distance the other reaches from the foot of the pole.

17. Show geometrically that in any triangle

$$\tan B = \frac{b \sin C}{c \cos B}, \text{ and } c^2 \cos^2 B + b^2 \sin^2 C = c^2.$$

Examine separately the cases in which the angle B is acute, right and obtuse.

18. Show geometrically that in any triangle $\tan B = \dfrac{c \sin B}{a - b \cos C}$ and deduce $\cot B + \cot C = \dfrac{a}{c} \operatorname{cosec} B$. Examine the effect of changing B from an acute to an obtuse angle. Give also the value of $\tan B$, using a different construction.

19. From the top of a cliff 100 feet high the angles of depression of two points on the horizontal plane below in a straight line with the point of observation are $30°$ and $15°$ respectively. Find how far the points are apart.

20. From the top of a tower the angles of depression of the top and the bottom of a flag-pole are 15° and 75° respectively. Show that the distance of the pole from the tower is a mean proportional between the height of the tower and the excess of its height above the pole.

21. Find the side of a regular octagon inscribed in a circle, radius r; also the perpendicular from the centre on the side, and thence the area of the polygon.

22. An isosceles triangle of wood is placed on the ground in a vertical position facing the sun. If $2a$ be the base of the triangle, b its height, and 30° the altitude of the sun, find the tangent of half the angle at the apex of the shadow. If this tangent be $\dfrac{a\sqrt{2}}{2b}$, find the altitude of the sun.

23. Standing straight in front of a house and opposite one corner, I find that its length subtends an angle whose sine is $\frac{2}{5}\sqrt{5}$, while its height, 51 feet, subtends an angle whose tangent is $\frac{3}{5}$. Find the length of the house.

24. The angular elevation of the top of a tree on the bank of a river from a point on the other bank directly opposite is α, while from a point at a distance d from the former and parallel to the stream, the elevation is β. Find the breadth of the river and the height of the tree.

25. At noon the altitude of the sun is 45°, and the shadow of a tree standing vertical on a hillside sloping to the north at an angle of 15° is 100 feet. Find the height of the tree.

26. The sides of a triangle are $m+n$, $m-n$, and $\sqrt{2(m^2+n^2)}$, the sine of one angle is $\dfrac{\sqrt{5}-1}{4}$. Find the other angles.

27. Prove that the area of a triangle right-angled at C is $s(s-c)$ where s is half the sum of the sides; and that if $s(s-c) = (s-a)(s-b)$ the triangle is right-angled.

28. In a triangle right-angled at C, AD is drawn to meet BC, making the angle BAD equal to the angle B. Show that

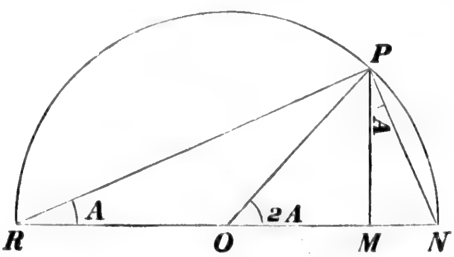
$$BD = \frac{c^2}{2a} \text{ and thence that } \tan(A - B) = \frac{(a-b)(a+b)}{2ab}.$$

29. If a, b, c are the lengths of three straight lines drawn from a point making equal angles with one another, and if straight lines be drawn joining their extremities, the area of the whole triangle thus formed is $\frac{\sqrt{3}}{4}(ab + bc + ca)$.

30. When the sun is in the east and 30° above the horizon, find the length of the shadow of a pole 20 feet long which is inclined to the south at an angle of 15° from the vertical. Find also its length when the sun is in the south and 60° above the horizon.

Ratios for Half and Double Angles.

73. *Given the ratios for any angle to find the ratios for half that angle; and conversely.*



At the centre O of a semicircle NPR, whose radius is a unit, make the angle NOP equal to the given angle, draw PM perpendicular to RN and join RP, PN. Then RPN is a right angle, and PRN is half the given angle. Euc. III., 31.

Denote NOP by $2A$, then angle $PRN = MPN = A$.

Then since $\qquad RO = OP = 1$

therefore $\qquad MP = \sin 2A$ and $OM = \cos 2A$

and $$\tan A = \frac{MP}{RM} = \frac{MP}{RO + OM} = \frac{\sin 2A}{1 + \cos 2A}. \qquad (1)$$

From (1) $\tan^2 A = \dfrac{\sin^2 2A}{(1+\cos 2A)^2} = \dfrac{1-\cos^2 2A}{(1+\cos 2A)^2} = \dfrac{1-\cos 2A}{1+\cos 2A}$. (2)

Then $\sec^2 A = 1 + \tan^2 A \quad = 1 + \dfrac{1-\cos 2A}{1+\cos 2A} = \dfrac{2}{1+\cos 2A}$

or
$$\cos^2 A = \frac{1+\cos 2A}{2}. \tag{3}$$

Similarly
$$\sin^2 A = \frac{1-\cos 2A}{2}. \tag{4}$$

Subtracting (4) from (3)
$$\cos 2A = \cos^2 A - \sin^2 A. \tag{5}$$

Rearranging (3) and (4)
$$\cos 2A = 2\cos^2 A - 1 \tag{6}$$
$$= 1 - 2\sin^2 A. \tag{7}$$

From (1) $\quad \sin 2A = (1+\cos 2A)\tan A$

and from (3) $\quad = 2\cos^2 A . \dfrac{\sin A}{\cos A}$

$$= 2\sin A \cos A. \tag{8}$$

Then $\quad \tan 2A = \dfrac{\sin 2A}{\cos 2A} = \dfrac{2\sin A\cos A}{\cos^2 A - \sin^2 A} = \dfrac{2\tan A}{1-\tan^2 A}$.

The last step being obtained by dividing numerator and denominator by $\cos^2 A$. Each of the preceding results should be memorized.

74. It will be instructive to obtain the preceding formulæ direct from the figure. The radius being unity we have

$$\frac{PM}{RM} = \tan A = \frac{MN}{PM}.$$

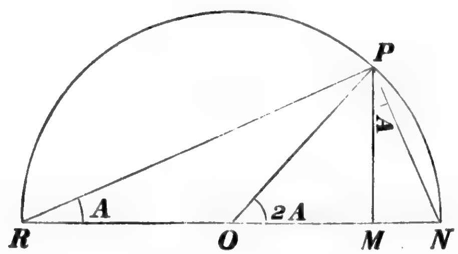
Therefore
$$\tan^2 A = \frac{PM}{RM}.\frac{MN}{PM} = \frac{MN}{RM} \tag{1}$$
$$= \frac{ON - OM}{RO + OM} = \frac{1-\cos 2A}{1+\cos 2A}. \tag{2}$$

Expressing the lengths of the various lines in terms of the radius and the ratios of the angle A, we have

$$RM = RP\cos A = (RN\cos A)\cos A = 2\cos^2 A, \qquad (3)$$
$$MN = PN\sin A = (RN\sin A)\sin A = 2\sin^2 A, \qquad (4)$$
$$MP = PN\cos A = (RN\sin A)\cos A = 2\sin A\cos A. \qquad (5)$$

Then $$RM = RO + OM$$
or, from (3) $$2\cos^2 A = 1 + \cos 2A. \qquad (6)$$

And $$MN = ON - OM$$
or, from (4) $$2\sin^2 A = 1 - \cos 2A. \qquad (7)$$



Again expressing OM successively in three different forms, we have

$$OM = \tfrac{1}{2}(RM - MN) = RM - RO = ON - MN,$$

from which $$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A. \qquad (8)$$

And $$PM = RP\sin A$$
$$= RN\sin A\cos A$$
or $$\sin 2A = 2\sin A\cos A. \qquad (9)$$

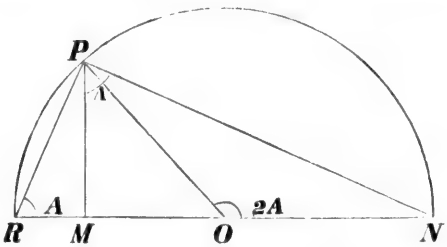
$\tan 2A = \frac{PM}{OM} = 2PM$

Also $$\tan 2A = \frac{2\,MP}{2\,OM} = \frac{2\,MP}{RM - MN}$$

from (1) $$= \frac{\dfrac{2\,MP}{RM}}{1 - \dfrac{MN}{RM}} = \frac{2 \tan A}{1 - \tan^2 A}. \qquad (10)$$

$\sin^2 A$

Similarly various other formulæ may be obtained.



As an additional exercise we give a second diagram in which the angle $2A$ is obtuse. It will be found that the preceding proof applies without change, simply remembering that in this case OM is negative.

75. The truth of the preceding demonstrations does not depend upon the symbols employed to denote the angles. Thus, we might have denoted the angle NOP by A, then NRP would have been $\frac{A}{2}$, and our results would have appeared in a slightly different form.

Thus $$\tan \frac{A}{2} = \frac{\sin A}{1 + \cos A}$$

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}, \text{ etc.}$$

The student should be familiar with each form. The formulæ are true for all values of the angle, though the proof given evidently restricts the angle NOP to be less than two right angles. Complete proofs will be given hereafter.

Ex. 1.—Prove $\dfrac{4\tan A\,(1-\tan^2 A)}{(1+\tan^2 A)^2}=\sin 4A$.

$$\frac{4\tan A\,(1-\tan^2 A)}{(1+\tan^2 A)^2}=\frac{4\sin A}{\cos A}\cdot\frac{1-\tan^2 A}{\sec^4 A}$$
$$=4\sin A\cos^3 A\left(1-\frac{\sin^2 A}{\cos^2 A}\right)$$
$$=4\sin A\cos A\,(\cos^2 A-\sin^2 A)$$
$$=2\sin 2A.\cos 2A$$
$$=\sin 4A.$$

Ex. 2.—Given $\operatorname{cosec}\theta-\sin\theta=\tan\dfrac{\theta}{2}$; find $\cos\theta$.

We have $$\frac{1}{\sin\theta}-\sin\theta=\frac{\sin\theta}{1+\cos\theta}\qquad\text{Art. 73. (1)}$$

or $$1-\sin^2\theta=\frac{\sin^2\theta}{1+\cos\theta}$$

therefore $$\cos^2\theta=\frac{1-\cos^2\theta}{1+\cos\theta}$$
$$=1-\cos\theta$$

or $$\cos^2\theta+\cos\theta=1$$

from which $$\cos\theta=\frac{\sqrt{5}-1}{2}.$$

EXERCISE X.

Prove the following identities:

1. $\sin 2A\cot A=1+\cos 2A$.
2. $\cos^4 A-\sin^4 A=\cos 2A$.
3. $\cot A+\tan A=2\operatorname{cosec} 2A$.
4. $\cot A-\tan A=2\cot 2A$.
5. $\operatorname{cosec} 2A+\cot 2A=\cot A$.
6. $\operatorname{cosec} 2A-\cot 2A=\tan A$.
7. $\sin 2A=\dfrac{2\tan A}{1+\tan^2 A}$.
8. $\sec 2A=\dfrac{\sec^2 A}{2-\sec^2 A}$.
9. $\dfrac{1}{1-\tan A}-\dfrac{1}{1+\tan A}=\tan 2A$.
10. $4\operatorname{cosec}^2 2\theta=\dfrac{\operatorname{cosec}^4\theta}{\operatorname{cosec}^2\theta-1}$.
11. $\dfrac{\cos A+\sin A}{\cos A-\sin A}=\tan 2A+\sec 2A$.

12. $\dfrac{\cot\theta + \tan\theta}{\cot\theta - \tan\theta} = \sec 2\theta.$ 13. $\dfrac{1+\tan\theta}{1-\tan\theta} = \sec 2\theta + \tan 2\theta.$

14. $\dfrac{2\sin A - \sin 2A}{2\sin A + \sin 2A} = \tan^2\dfrac{A}{2}.$ 15. $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\dfrac{\theta}{2}.$

16. $\sin^3\theta + \cos^3\theta = (\sin\theta + \cos\theta)(1 - \frac{1}{2}\sin 2\theta).$

17. $\sin\theta(\sec\theta + \operatorname{cosec}\theta)(1 - \tan\theta) = 2\tan\theta\cot 2\theta.$

18. $\sin^2 A - \cos^2 A\cos 2B = \sin^2 B - \cos 2A\cos^2 B.$

19. $2\sin^2 A\sin^2 B + 2\cos^2 A\cos^2 B = 1 + \cos 2A\cos 2B.$

20. $\operatorname{cosec} 2A + \cot 4A = \cot A - \operatorname{cosec} 4A.$

21. $\operatorname{cosec} 4A(2 + 2\cos 2A + 3\cos 4A) = \cot A + \cot 2A + \cot 4A.$

22. In a triangle, right-angled at C, show that

$$\tan\frac{A}{2} = \frac{a}{b+c} = \frac{c-b}{a}, \quad \sin\frac{A}{2} = \sqrt{\frac{c-b}{2c}}.$$

23. If $\tan\theta = \dfrac{b}{a}$, show that $a\cos 2\theta + b\sin 2\theta = a.$

24. If $\tan 2A = \dfrac{a}{b}$, find $\tan A$.

25. Prove geometrically $\cot A = \dfrac{\sin 2A}{1-\cos 2A} = \dfrac{\cot^2 A - 1}{2\cot 2A}.$

26. If $\cos\theta = \dfrac{a\cos\phi - b}{a - b\cos\phi}$, show that $\tan^2\dfrac{\theta}{2} = \dfrac{a+b}{a-b}\tan^2\dfrac{\phi}{2}.$

27. Show geometrically that the area of any triangle is $\frac{1}{4}(a^2\sin 2B + b^2\sin 2A)$. Examine the formula for acute and for obtuse angled triangles separately.

28. The line AD is drawn bisecting the angle BAC, and BC, BD, are drawn perpendicular to AC and AD respectively. Show that $BA.BC = 2BD.AD$ and $BA.AC = AD^2 - BD^2$.

29. Find the perimeter of a polygon of n sides (1) inscribed in a circle, (2) described about a circle, whose radius is r.

30. Find the area of each of the polygons in the preceding example.

31. From the consideration that each of the polygons in the previous example becomes a circle when $n = \infty$, show that when θ is the circular measure of an indefinitely small angle

$$\frac{\sin \theta}{\theta} = \frac{\tan \theta}{\theta} = 1.$$

32. Show that the area of a polygon of $2n$ sides inscribed in a circle is a mean proportional between the inscribed and the circumscribed polygons of n sides.

33. A flag-staff a feet high stands on the top of a tower b feet high. At what distance from the base on a level plain will staff and tower subtend equal angles?

34. A person in line with two towers, and at distances of 100 and 150 yards from them, observes that their apparent altitudes are the same; he then walks towards them a distance of 60 yards and finds that the angle of elevation of the nearer is just double that of the more distant. Find the heights of the towers.

35. A tower of height h and a spire which surmounts it are each observed to subtend an angle α at a point on the horizontal plane. Show that the height of the spire is $h \sec 2\alpha$.

36. The angle of elevation of a tower is observed; at a point a feet nearer, the elevation is the complement of the former; b feet nearer still, it is double the first elevation. Show that the height of the tower is $\frac{1}{2}\sqrt{(a+2b)(3a+2b)}$.

CHAPTER V.

SOLUTION OF TRIANGLES.

76. In practical applications of trigonometry two processes are constantly required.

1. Having given the magnitude of an angle, we require the numerical value of one or more of its trigonometrical ratios.

2. Having given the numerical value of one of the trigonometrical ratios of an angle, we require the magnitude of the angle.

To facilitate these processes we have given in the Appendix the values of the ratios for angles between 0° and 45°, at intervals of $10'$. From these we can obtain the ratios for any given angle, or the angle which corresponds to any given ratio, by methods which we proceed to explain.

77. The trigonometrical ratios are functions of the angle, *i.e.*, they are quantities whose values depend upon the magnitude of the angle and change as the angle changes. In this connection we shall assume the truth of the following general principle:

The change in value of the function of a variable is approximately proportional to the change in the variable, providing the change be sufficiently small.

This is known as "The Principle of Proportional Parts." It is also sometimes called "The Rule of Proportional Differences."

The following examples will show clearly the meaning and application of this principle:

Ex. 1.—Given the values of sin 13° 10′, and sin 13° 20″, to find the value of sin 13° 14′ 25″.

From the tables we find

	sin 13° 20′	= 23062
	sin 13° 10′	= 22778
from which	difference for 10′	= 284
therefore,	difference for 1′	= 28.4
and	difference for 4′ 25″	= $28.4 \times 4\frac{25}{60}$
		= 125.4.

Then to	sin 13° 10′	= 22778
add difference for	4′ 25″	= 125
therefore,	sin 13° 14′ 25″	= .22903.

Ex. 2.—Find the value of cos 42° 13′ 14″.

From the tables we find

	cos 42° 10′	= 74120
	cos 42° 20′	= 73924
from which	difference for 10′	= 196
therefore,	difference for 1′	= 19.6
and	difference for 3′ 14″	= $19.6 \times 3\frac{14}{60}$
		= 63.89.

Now, as the angle *increases* the cosine *diminishes*,

therefore from	cos 42° 10′	= 74120
take	difference for 3′ 14″	= 64
Therefore	cos 42° 13′ 14″	= .74056.

Ex. 3.—Find the angle whose tangent is .78439.

From the tables we have

	tan 38° 10′	= 78622
	tan 38°	= 78129
from which	difference for 10′	= 493.

Again given tangent $= 78439$
tan $38^\circ = \underline{78129}$
difference for increase of angle $= 310$
But $10'$ gives a difference of 493
therefore, 310 is the difference for $\frac{310}{493}$ of $10' = 6'\ 17''$
and the required angle is $38^\circ\ 6'\ 17''$.

Ex. 4.—Find angle whose cotangent is 2.1347.

From the tables we find

$$\begin{aligned} \cot 25^\circ \quad &= 2.1445 \\ \cot 25^\circ\ 10' &= \underline{2.1283} \end{aligned}$$

from which difference for $10' = 162$

Again cot $25^\circ = 2.1445$
given cot $= \underline{2.1347}$
difference for increase of angle $= 98.$

But $10'$ gives a difference of 162
therefore 98 is the difference, for $\frac{98}{162}$ of $10' = 6'\ 3''$ nearly,
and the required angle is $25^\circ\ 6'\ 3''$.

In this example, carefully observe that as the angle increases the cotangent diminishes, and since the given cotangent is less than cot 25°, the required angle is greater than 25°. Compare Ex. 2.

78. The values of the ratios of angles greater than 45° may easily be deduced from those of angles less than 45°, by using their complementary or supplementary values.

Thus, $\sin 70^\circ = \cos 20^\circ$, $\sin 125^\circ = \sin 55^\circ = \cos 35^\circ$,
$\tan 115^\circ = -\tan 65^\circ = -\cot 25^\circ$, etc.

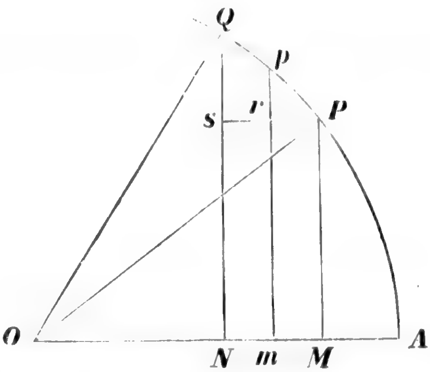
The values of the secants and cosecants have not been given, since they are not often used. Their values may be calculated

directly from those of the cosine and sine, but more easily from the formula

$$\operatorname{cosec} A = \tfrac{1}{2}\left\{\tan\frac{A}{2} + \cot\frac{A}{2}\right\}$$

which gives the value of the cosecant. The secant may be found by calculating the cosecant of its complement.

79. The extent to which the principle of proportional parts can be safely employed depends upon the nature of the function to which it is applied. In the case of the sine and cosine an instructive geometrical illustration can easily be given.



Let AOP and AOQ be two angles whose sines are given, and AOp an angle lying between them whose sine is required. The radius of the circle being unity, we may be supposed to know the lengths of PM and QN, and to require the length of pm. Now, *assuming PQ to be a straight line*, from the similar triangles Ppr, PQs, we have

$$\frac{pr}{Pp} = \frac{Qs}{PQ}, \text{ or } pr = \frac{Pp}{PQ} \cdot Qs.$$

The length of pr thus found added to PM gives the length of pm required.

The length of Pr may be similarly found and *subtracted* from OM for the cosine of AOp.

In the preceding examples POQ represents an angle of $10'$, and since a complete revolution, or $360°$, contains 2160 such angles, we have in effect treated the circle as though it were a regular polygon of 2160 sides of which PQ is one. The error arising from this assumption is too small to affect the results so far as five decimal places are concerned.

EXERCISE XI.

1. Find $\sin 14° \, 15' \, 20''$, and the angle whose sine is .37895.

2. Find $\cos 35° \, 14' \, 30''$, and the angle whose cosine is .24786.

3. Find $\tan 39° \, 15' \, 45''$, and the angle whose tangent is .78926.

4. Find $\cot 43° \, 14' \, 17''$, and the angle whose cotangent is 2.6784.

5. Find the sine and the cosine of $125° \, 13'$.

6. Find $\tan 78° \, 14' \, 37''$, and the angle whose tangent is 6.2378.

7. Find the angles of a triangle whose sides are 3, 4, 5.

8. Find the obtuse angle whose sine is .87253.

9. Find the tangent of the angle in the preceding example.

10. The cosine of an angle is $-.38247$; find the angle and thence its sine.

11. Given $\sin(A+B) = \frac{2}{3}$, and $\cos(A-B) = \frac{3}{4}$, in which A and B are the angles of a triangle, A being the greater. Find A and B, giving two solutions.

12. Given $\tan \frac{1}{2}(A-B) = \dfrac{a-b}{a+b} \cot \dfrac{C}{2}$, and $A + B = 105° \, 40'$, $a = 5$, $b = 3$; find A, B and C, which are the angles of a triangle.

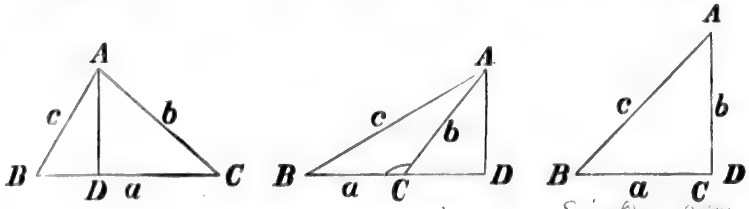
Sides and Angles of a Triangle.

80. *The sides of any triangle are proportional to the sines of the opposite angles; or in symbols*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

B a C

Let ABC be any triangle, and from A draw AD at right angles to BC, or to BC produced, and denote the sides opposite to the angles A, B, C by a, b, c respectively.



Then $$b \sin C = AD = c \sin B$$

from which $$\frac{b}{\sin B} = \frac{c}{\sin C}.$$

Similarly, by drawing a perpendicular from B it may be shown that

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

and therefore $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

The above is known as "The Sine Rule."

81. *To express the base of a triangle in terms of the sides and the base angles.*

With the construction of Art. 80 we have :

In each figure $c \cos B$ = length of BD taken positively
in Fig. 1 $b \cos C$ = length of CD " "
in Fig. 2 $b \cos C$ = length of CD taken negatively
in Fig. 3 $b \cos C = 0$.

Therefore in all cases

$$BC = BD + DC \text{ (including direction)}$$

or $$a = c \cos B + b \cos C. \qquad (1)$$
Similarly $$b = a \cos C + c \cos \cancel{B}\ A \qquad (2)$$
and $$c = b \cos A + a \cos \cancel{C}\ B \qquad (3)$$

82. *To express the cosine of an angle of a triangle in terms of the sides.*

In Fig. 1	$AB^2 = BC^2 + CA^2 - 2BC \,.\, CD.$	Euc. II., 13.
In Fig. 2	$AB^2 = BC^2 + CA^2 + 2BC \,.\, CD.$	Euc. II., 12.
In Fig. 3	$AB^2 = BC^2 + CA^2.$	Euc. I., 47.

Now, in Fig. 2, $b \cos C$ = length of CD taken negatively, and in Fig. 3, $\quad b \cos C = 0.$

Therefore in all cases

$$c^2 = a^2 + b^2 - 2ab \cos C$$

or
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}. \qquad (1)$$

The values of $\cos A$ and $\cos B$ may now be written from symmetry.

Thus,
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \ \cos B = \frac{c^2 + a^2 - b^2}{2ca}. \qquad (2)$$

This may be quoted as "The Cosine Rule."

83. *To express the sine, cosine and tangent of half an angle of a triangle in terms of the sides.*

Let s denote half the sum of the sides of the triangle, so that

$$a + b + c = 2s, \ b + c - a = 2(s - a)$$
$$c + a - b = 2(s - b), \ a + b - c = 2(s - c).$$

Then
$$2 \sin^2 \frac{A}{2} = 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 - (b - c)^2}{2bc}$$
$$= \frac{(a - b + c)(a + b - c)}{2bc} = \frac{2(s - b)(s - c)}{bc},$$

or
$$\sin \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}}. \qquad (1)$$

Again, $$2\cos^2\frac{A}{2}=1+\cos A=1+\frac{b^2+c^2-a^2}{2bc}=\frac{(b+c)^2-a^2}{2bc}$$

$$=\frac{(b+c+a)(b+c-a)}{2bc}=\frac{2s(s-a)}{bc}$$

or $$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}. \qquad (2)$$

From equations (1) and (2) we get

$$\tan\frac{A}{2}=\sin\frac{A}{2}\div\cos\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}. \qquad (3)$$

Equations (1), (2) and (3) may be quoted as "The Half-angle Formulæ."

84. *To express the sine of an angle of a triangle in terms of the sides.*

From equations (1) and (2), Art. 83, we get

$$\sin A=2\sin\frac{A}{2}\cos\frac{A}{2}=2\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{s(s-a)}{bc}}$$

$$=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} \qquad (4)$$

Similarly the values of sin B and sin C may be written. These results may also be easily obtained direct from the values of cos A, cos B and cos C, given in Art. 82.

85. *In any triangle to prove* $\dfrac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\dfrac{a-b}{a+b}$, *assuming that* A *is greater than* B, *and consequently that* a *is greater than* b.

Let ABC be the given triangle.

With centre C and radius CA, describe a semicircle cutting BC in D and BC produced in E. Join EA, AD, and draw DF parallel to EA to meet AB in F.

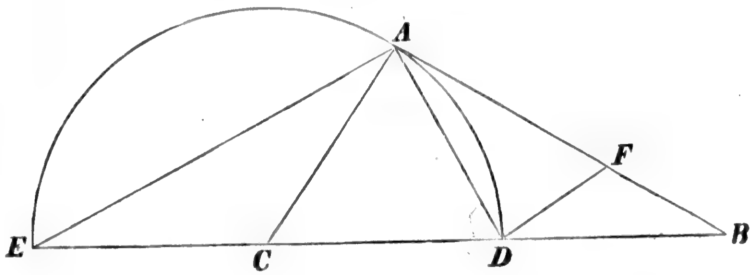
Then angles EAD, ADF are right angles, Euc. III., 31.
and triangles BFD, BAE, have their sides proportional.
Euc. VI., 2.

Now $\qquad$ angle $ADE = \frac{1}{2}$ angle ACE $\qquad$ Euc. III., 20.

$= \frac{1}{2}(A+B)$, $\qquad$ Euc. I., 16.

and $\qquad$ angle $DAF = ADE - ABD$ $\qquad$ Euc. I., 32.

$$= \tfrac{1}{2}(A+B) - B$$
$$= \tfrac{1}{2}(A-B).$$



Also $\qquad BD = BC - DC \qquad$ and $BE = BC + CE$
$\qquad = a - b \qquad\qquad = a + b.$

Then
$$\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} = \frac{\tan DAF}{\tan ADE} = \frac{DF}{AD} \div \frac{AE}{AD}$$
$$= \frac{DF}{AE} = \frac{BD}{BE} = \frac{a-b}{a+b}. \qquad \text{Euc. VI., 4.}$$

This is often quoted as "The Sum and Difference Formula."

86. *To find the area of a triangle.*

In the figure of Art. 80 we have

Area of triangle $ABC = \frac{1}{2} . BC . AD$

$$= \tfrac{1}{2} . ab \sin C, \qquad (1)$$

or in words, *the area of a triangle is equal to half the product of any two sides and the sine of the angle between them.*

If for sin C we substitute its value from Art. 84, we get

$$\text{Area of triangle } ABC = \tfrac{1}{2} ab . \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)} \qquad (2)$$

which gives the area in terms of the sides alone. This latter expression is frequently denoted by S.

Again from $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

we get $\qquad a = \dfrac{c \sin A}{\sin C}, \quad b = \dfrac{c \sin B}{\sin C}.$

Substituting these values in (1) we get

$$\text{area of triangle } ABC = \frac{c^2 \sin A \sin B}{2 \sin C},$$

which gives the area in terms of the angles and one side.

87. We purposely refrain for the present from giving any rules for the application of the preceding formulæ. The subject will be treated in a future chapter. In the meantime we give a few examples.

Ex. 1.—Given $A = 45°$, $B = 60°$, $c = 10$, to find a, b and C.

From $A + B + C = 180°$, we get at once $C = 75°$.
Then from the Sine Rule

$$\frac{a}{\sin 45°} = \frac{b}{\sin 60°} = \frac{10}{\sin 75°}$$

from which

$$a = \frac{10 \sin 45°}{\sin 75°} = \frac{10}{\sqrt{2}} \cdot \frac{4}{\sqrt{6}+\sqrt{2}} = 10(\sqrt{3}-1) = 7.3205.$$

Similarly, $b = 5\sqrt{6}(\sqrt{3}-1) = 8.96576.$

The values of the ratios might have been taken from the tables and the introduction of surds avoided.

Ex. 2.—Given $a = 10$, $b = 6$, $C = 37° \ 10' \ 15''$, to find A, B and c. We have

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cos C \\ &= 100 + 36 - 120 \times .79684 \\ &= 40.3792 \end{aligned}$$

or

$$c = 6.35446.$$

Then from the Sine Rule

$$\sin A = \frac{a \sin C}{c} \qquad\qquad \sin B = \frac{b \sin C}{c}$$

$$= .95082 \qquad\qquad = .57049$$

or

$$A = 71° \ 57' \ 20'' \qquad\qquad B = 34° \ 47' \ 5''.$$

The above are the acute angles whose sines satisfy the given equations, but in each case there is also an obtuse angle which has the same sine. To determine which must be taken we observe that since $a > b \therefore A > B$, and consequently B is acute. And since $A + B + C = 180°$ we shall find that A must be obtuse.

Therefore $A = 180° - (71° \ 57' \ 20'') = 108° \ 2' \ 40''$, $B = 34° \ 47' \ 5''$, are the values required.

Another method of solution might have been adopted.

Thus

$$\begin{aligned} \tan \frac{A - B}{2} &= \frac{a - b}{a + b} \cot \frac{C}{2} \\ &= \tfrac{1}{4} \cot 18° \ 35' \ 7\tfrac{1}{2}'' \\ &= .74350, \end{aligned}$$

from which

$$\frac{A - B}{2} = 36° \ 37' \ 51''$$

and

$$\frac{A + B}{2} = 71° \ 24' \ 52'' = 90° - \frac{C}{2}$$

therefore

$$A = 108° \ 2' \ 43'', \text{ and } B \ 34° \ 47' \ 2''.$$

It will be observed that there is a discrepancy of $3''$ between the two solutions. This arises from the limited number of decimal places used. Whilst such an error could not be tolerated in very accurate work, it is still entirely inappreciable for ordinary purposes.

Having found the angles A and B, the side c may be found from the formula

$$c = a \cos B + b \cos A$$

which gives
$$\begin{aligned} c &= 10 \cos 34^\circ\ 47'\ 2'' + 6 \cos 108^\circ\ 2'\ 43'' \\ &= 8.2130 - 1.85862 \\ &= 6.35448. \end{aligned}$$

This result agrees with the former to five significant figures, a degree of accuracy much greater than is usually attained in practical work.

Ex. 3.—Given A, b, a, to construct the triangle and find B, C, c.

Make CAX equal to the given angle A, AC equal to b, and from C as centre with radius equal to a describe a circle; join C to the point, or points, if any, in which this circle cuts AX, and the result will be the construction required.

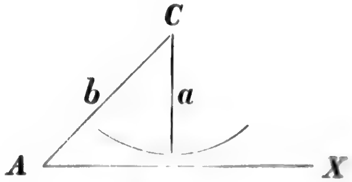


Fig. 1.

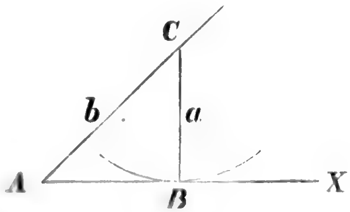


Fig. 2.

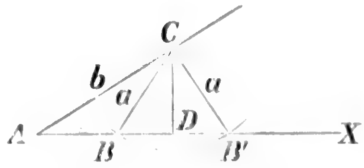


Fig. 3.

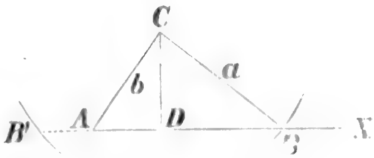


Fig. 4.

Different cases may arise, which we proceed to examine. The circle may not meet the line AX, may touch it, may cut it twice on the same side of A, or may cut it on opposite sides of A. The shortest line which can be drawn from C to meet AX is the

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perpendicular CD, which is equal to $b \sin A$. We have then the following results:

I.	$a < b \sin A$	(Fig. 1).	No solution possible.
II.	$a = b \sin A$	(Fig. 2).	One solution; triangle right-angled.
III.	$a > b \sin A$		
	(1) $a < b$	(Fig. 3).	Two solutions; CAB, CAB'.
	(2) $a = b$		One solution; triangle isosceles.
	(3) $a > b$	(Fig. 4).	One solution; CAB.

Thus, when two sides of a triangle and the angle opposite one of them are given two triangles may sometimes be drawn fulfilling the required conditions. This is consequently known as the ambiguous case in the solution of triangles.

88. The preceding results have been obtained from an inspection of the geometrical diagram. It will now be very instructive to examine the equations which connect the sides and angles of a triangle, and observe how each of these results is indicated by symbols. We shall do this in two ways.

From triangle CAB (or CAB'), Fig. 3, we have,

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Arranging this as a quadratic equation to find c, we have

$$c^2 - 2b \cos A . c + b^2 - a^2 = 0.$$

Solving,
$$c = b \cos A \pm \sqrt{b^2 \cos^2 A - b^2 + a^2}$$
$$= b \cos A \pm \sqrt{a^2 - b^2 \sin^2 A}.$$

Now, from Fig. 3, it will be readily observed that $CD = b \sin A$, $CB = a$, and consequently $BD = B'D = \sqrt{a^2 - b^2 \sin^2 A}$. The two values of c are therefore AB and AB', which is in harmony with the fact that the original equation belongs equally to the two triangles.

Again, from the Sine Rule

$$\sin B = \frac{b \sin A}{a}$$

from which B may be found, then C is known since two angles are known, and then c may also be found from the Sine Rule.

Now compare the three modes of investigation — the geometrical, the algebraical, and the trigonometrical:

I. $a < b \sin A$; the circle does not meet the line; the value of c is imaginary; $\sin B > 1$; impossibility denoted.

II. $a = b \sin A$; the circle meets the line in one point; the quadratic has equal roots; sine $B = 1$; a right-angled triangle.

III. $a > b \sin A$.

(1) $a < b$; the circle cuts the line twice on the same side of A; the quadratic has two real and positive roots; an acute and an obtuse angle each satisfy the value of $\sin B$; two solutions.

(2) $a = b$; the circle cuts the line once and passes through A; the quadratic has one root positive and one root zero; $B = A$; the triangle is isosceles.

(3) $a > b$; the circle cuts the line on opposite sides of A; one root of quadratic is positive and one negative; the *obtuse* value of B is inadmissible; one solution.

From the value of $\sin B$ alone we are unable to determine whether there are one or two solutions in the last two cases. The additional test required is furnished by Euc. I., 17. When $a = b$, $A = B$ and both angles must be acute; when $a > b$, $A > B$, $\therefore B$ must be acute, and in each case there is but one solution.

The angle A has been considered acute throughout the investigation. The student should examine the cases in which A is right or obtuse.

EXERCISE XII.

1. Given $A = 30°$, B 45°, $c = 20$; find a, b and C.

2. Given $A = 60°$, $b = 7$ feet, $c = 5$ yards; find a, $\sin B$ and $\sin C$.

3. The sides of a triangle are 5, 6, 7; find the cosine of the least angle, the sine of the greatest angle, and the area.

4. Given $a=7$, $b=8$, $c=9$; find $\sin\frac{A}{2}$, $\cos\frac{B}{2}$ and $\tan\frac{C}{2}$.

5. Two sides of a triangle are 13 and 15, and the cosine of the included angle is $\frac{33}{65}$; find the remaining side and the area.

6. The sides of a triangle are 21, 22 and 23; find all the angles.

7. The sides of a triangle are 13, 37 and 40; find the least angle and the perpendicular on the longest side from the opposite angle.

8. The two sides of a triangle are 8 and 10 inches respectively, and the included angle is $38° 15'$; find the remaining side and angles.

9. Two sides of a triangle are in the ratio of $2:10$, and the included angle is $15°$; find the remaining angles.

10. Given $a=2$, $b=\sqrt{6}$, $c=1+\sqrt{3}$; find A, B and C.

11. The sides of a triangle are x^2+x+1, $2x+1$ and x^2-1; find the cosine of the greatest angle and thence the angle itself.

12. If the angle A be acute and $\sin A=\frac{1}{3}$, $b=24$; find the least value of a with which a triangle can be formed. Find B when $a=16$.

13. Given $A=18°$, $a=4$, $b=4+\sqrt{80}$; find the remaining parts of the triangle.

14. Given $A=15°$, $a=4$, $b=4+\sqrt{48}$; find B, C and c.

15. Given $C=18°$, $a-c=2$, $ac=4$; find A and B.

16. The sides of a triangle are 8, 9 and 10; find the length of the line joining the largest angle to the centre of the opposite side.

17. The sides of a triangle are 25, 30 and 45; find the length of the bisector of the smallest angle, and the angle which the bisector makes with the base.

18. Two sides of a triangle are 20 and 32 rods respectively, and the area is one acre; find the third side.

19. The base angles of a triangle are $22\frac{1}{2}°$ and $112\frac{1}{2}°$; show that its base is twice its height.

20. Which, and how many of the sides and angles of a triangle must be known before the others can be found? State all the different cases which may occur, and refer each to a corresponding proposition in Euclid.

21. From the formulæ of Arts. 80 and 81, prove that in any triangle $\sin(B+C)=\sin B \cos C+\cos B \sin C$.

22. From the three equations of Art. 81, find the values of $\cos A$, $\cos B$ and $\cos C$, in terms of the sides.

23. If $\sin A=2\cos B\sin C$, show that the triangle is isosceles.

24. If $a\cos A=b\cos B$, the triangle is either right-angled or isosceles.

25. If $c^2=a^2+ab+b^2$, find $\cos C$, and thence show that the area of the triangle is $\dfrac{ab\sqrt{3}}{4}$.

26. Given $\sin C+\cos C=\dfrac{a}{b}$, find B.

27. Given the value of A, b and a, show that the sum of the areas of the two triangles which can be formed is $\frac{1}{2}b^2\sin 2A$, and the difference of their areas is $b\sin A\sqrt{a^2-b^2\sin^2 A}$.

28. In a triangle, CD is perpendicular to the base, and CE bisects the vertical angle; show that the angle $ECD=\frac{1}{2}(A-B)$, and thence that $\tan AEC=\dfrac{a+b}{a-b}\tan\dfrac{C}{2}$.

29. From the angle A of any triangle ABC a perpendicular is drawn to the base, and from D perpendiculars DE and DF are drawn to AB and AE. Show that

$$DE \,.\, \cos C=DF \,.\, \cos B \text{ and } AE \,.\, EB \,.\, \cos^2 C=AF \,.\, FC \,.\, \cos^2 B.$$

30. Given $(a+b+c)(b+c-a)=3bc$; find A.

31. If $b+c:c+a:a+b=4:5:6$, find A, and if the area be $60\sqrt{3}$, find a, b, c.

32. Three circles whose radii are a, b, c, touch each other externally; find the area of the triangle formed by joining their centres.

33. Each of two ships lying half a mile apart observes the angle subtended by the other ship and a fort. The angles are $56^\circ\ 19'$ and $63^\circ\ 14'$; find the distance of each ship from the fort.

34. From the top of a hill I observe two successive milestones in the plain below, and in a straight line before me, and find their angles of depression to be $5^\circ\ 30'$, $14^\circ\ 20'$; what is the height of the hill?

35. The angle of elevation of a tower 100 feet high, due north of an observer, was 50°; what will be its angle of elevation after the observer has walked due east 100 feet?

36. Prove that the perpendicular from C upon the opposite side of a triangle may be expressed by

$$\frac{a^2 \sin B + b^2 \sin A}{a+b}.$$

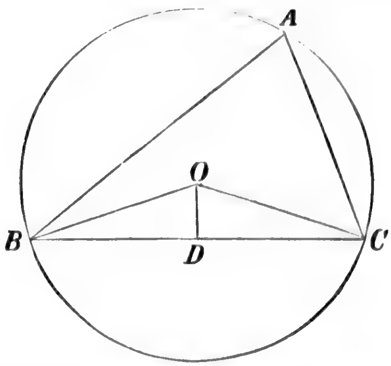
37. If perpendiculars be drawn from the angles of a triangle upon the opposite sides, show that the sides of the triangle formed by joining the feet of these perpendiculars are $a \cos A$, $b \cos B$ and $c \cos C$.

CHAPTER VI.

PROPERTIES OF CIRCLES, TRIANGLES AND POLYGONS.

89. In this chapter will be given a few of the simpler propositions relating to circles, triangles and polygons.

90. *To find the radius of the circle described about a given triangle.*



Let ABC be the given triangle, O the centre of the circle described about it. Draw OD perpendicular to BC and denote the radius OB by R. Then

$BD = \frac{1}{2}\,BC = \frac{a}{2}$, and angle $BOD = \frac{1}{2}\,BOC = A$. Euc. III., 20.

Then $\qquad OB \sin BOD = BD$, i.e., $R \sin A = \frac{a}{2}$,

or,
$$R = \frac{a}{2 \sin A} \tag{1}$$

which gives R in terms of a side and the opposite angle.

Again
$$R = \frac{a}{2 \sin A} = \frac{abc}{2bc \sin A} = \frac{abc}{4S} \tag{2}$$

which gives R in terms of the sides alone.

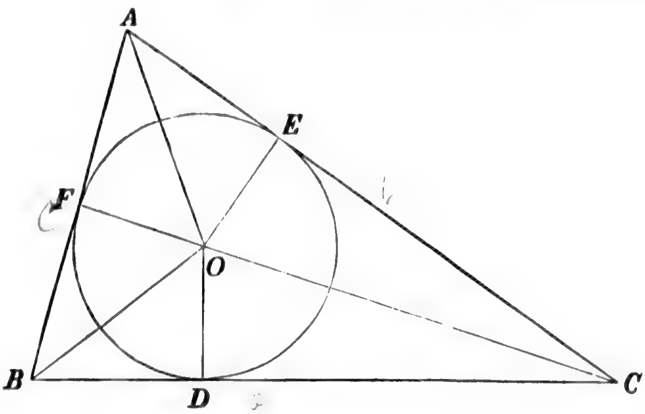
Also from equation (1) we get at once

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \qquad (3)$$

which gives another proof of Art. 80. Also from $\frac{a}{2R} = \sin A$, we get the important theorem:

The ratio of any chord in a circle to the diameter is equal to the sine of the angle at the circumference subtended by the chord.

91. *To find the radius of the circle inscribed in a given triangle.*



Let ABC be any triangle, O the centre of the inscribed circle touching the sides in the points D, E, F; then OD, OE, OF are perpendicular to BC, CA, AB, and OA, OB, OC bisect the angles A, B, C. Euc. IV., 4.

Denote the radius by r, and the area of the triangle by S.

Then
$$\begin{aligned} S &= \text{sum of areas of } BOC,\ COA,\ AOB \\ &= \tfrac{1}{2}(OD \cdot BC + OE \cdot CA + OF \cdot AB) \\ &= \tfrac{1}{2}(ra + rb + rc) \\ &= \frac{r}{2}(a + b + c) = rs. \end{aligned}$$

Therefore $$r = \frac{S}{s}. \quad = \frac{\Delta}{S} \qquad (1)$$

Again, since $AF = AE,\ BD = BF,\ CD = CE$

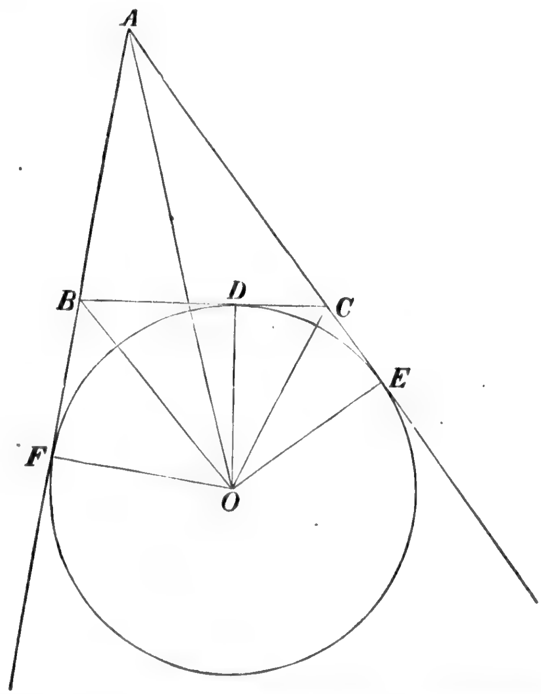
therefore $AF + BD + CD = \frac{1}{2}(a + b + c) = s,$

from which $AF = s - a.$

Then $$r = OF = AF \tan OAF = (s - a) \tan \frac{A}{2}. \qquad (2)$$

92. An **escribed circle** of a triangle is a circle which touches one side of the triangle and the other two sides produced.

93. *To find the radius of an escribed circle of a triangle*



Let ABC be any triangle, O the centre of the escribed circle touching the side BC, and the sides AC and AB produced, in D, E, F, then OD, OE, OF are perpendicular to BC, CA, AB, and OA, OB, OC bisect the angle A and the exterior angles at

B and C respectively. Denote the radius by r_1, and the area of the triangle by S. Then

$$\begin{aligned} S &= \text{sum of areas of } AOB,\ AOC \text{ less the area } BOC \\ &= \tfrac{1}{2}(OF\,.\,AB + OE\,.\,AC - OD\,.\,BC) \\ &= \tfrac{1}{2}(r_1 c + r_1 b - r_1 a) \\ &= \frac{r_1}{2}(b + c - a) = r_1(s - a), \end{aligned}$$

$$\therefore \qquad r_1 = \frac{S}{s - a}. \qquad (1)$$

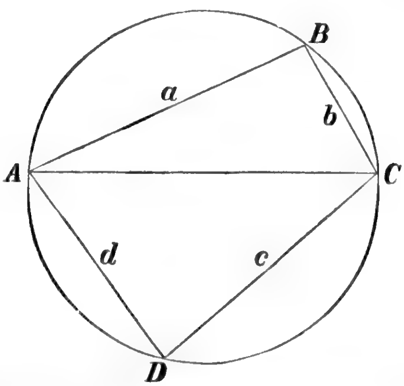
Again, since $2AF = AF + AE = (AB + BD) + (AC + CD)$
$= AB + BC + CA = 2s$

therefore $AF = s$.

Then $r_1 = OF = AF \tan\dfrac{A}{2} = s \tan\dfrac{A}{2}$ (2)

similarly, $r_2 = s \tan\dfrac{B}{2}$ and $r_3 = s \tan\dfrac{C}{2}$. (3)

94. *To find the cosine and the sine of an angle of a quadrilateral inscribed in a circle in terms of its sides.*



Let $ABCD$ be the quadrilateral; denote its sides by a, b, c, d, as in the figure, and let $a + b + c + d = 2s$, and consequently $a + b + c - d = 2(s - d)$, $a + b - c + d = 2(s - c)$, etc.

From the triangles ABC and ADC,

$$AC^2 = a^2 + b^2 - 2ab \cos B \qquad \text{Art 82.}$$
$$= c^2 + d^2 - 2cd \cos D$$

and since the angles B and D are supplementary, Euc. III., 22.

$$\cos D = -\cos B.$$

Therefore $\quad a^2 + b^2 - 2ab \cos B = c^2 + d^2 + 2cd \cos B$

from which
$$\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)} \qquad (1)$$

and thence
$$\sin^2 B = 1 - \frac{(a^2 + b^2 - c^2 - d^2)^2}{4(ab + cd)^2}.$$

Factoring this expression and substituting, we easily obtain

$$\sin B = \frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{ab + cd}. \qquad (2)$$

95. *To find the area of a quadrilateral which can be inscribed in a circle, and the radius of the circle circumscribing a quadrilateral, in terms of the sides.*

From the figure of the preceding article we have

Area of $ABCD = \frac{1}{2}(ab \sin B + cd \sin D)$ Art. 86.

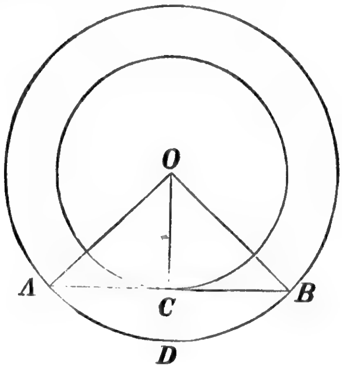
$$= \frac{1}{2}(ab + cd) \sin B, \text{ since } \sin B = \sin D$$
$$= \sqrt{\cancel{2}(s-a)(s-b)(s-c)(s-d)} \qquad \text{Art. 94 (2)}$$

From triangle ABC

$$AC^2 = a^2 + b^2 - 2ab \cos B \qquad \text{Art. 82.}$$
$$= a^2 + b^2 - \frac{2ab(a^2 + b^2 - c^2 - d^2)}{2(ab + cd)} \qquad \text{Art. 94 (1)}$$
$$= \frac{(ac + bd)(ad + bc)}{ab + cd}$$

Then $$R = \frac{AC}{2 \sin B} = \frac{1}{4}\sqrt{\frac{(ab + cd)(ac + bd)(ad + bc)}{(s-a)(s-b)(s-c)(s-d)}}. \qquad \text{Art. 90.}$$

96. *To find the radii of the inscribed and circumscribed circle of a regular polygon of any number of sides.*



Let AB be the side of a regular polygon of n sides; let O be the centre of each of the circles, OC the radius of the inscribed circle, OA the radius of the circumscribed circle. Let $AB=a$, $OA=R$, $OC=r$.

The angle AOB being the nth part of four right angles, we have

$$AOB=\frac{2\pi}{n}, \quad \therefore\ AOC=\frac{\pi}{n}$$

and
$$R\sin\frac{\pi}{n}=AC=\frac{a}{2}=r\tan\frac{\pi}{n}.$$

Therefore
$$R=\frac{a}{2\sin\frac{\pi}{n}}, \text{ and } r=\frac{a}{2\tan\frac{\pi}{n}}.$$

97. *To find the area of a regular polygon of* n *sides.*

With the figure and notation of the preceding article we have

$$\text{Area}=n.AC.OC=n.\frac{a}{2}.\frac{a}{2}\cot\frac{\pi}{n}=\frac{na^2}{4}\cot\frac{\pi}{n}. \tag{1}$$

or
$$=n.r\tan\frac{\pi}{n}.r=nr^2\tan\frac{\pi}{n}. \tag{2}$$

or
$$=n.R\sin\frac{\pi}{n}.R\cos\frac{\pi}{n}=\frac{n}{2}R^2\sin\frac{2\pi}{n}. \tag{3}$$

Equations (1), (2) and (3) give the area in terms of a side, the radius of the inscribed circle, and the radius of the circumscribing circle respectively.

98. *To find the area of a circle.*

Describe a polygon of n sides about the circle. (Fig. Art. 96).

$$\begin{aligned} \text{Perimeter} &= n.AB, & \text{area} &= n.AOB, \\ &= 2n.AC, & &= n.AC.OC, \\ &= 2nr\tan\frac{\pi}{n}, & &= nr^2\tan\frac{\pi}{n}. \end{aligned}$$

Making n infinitely great the polygon becomes a circle. Equating the perimeter of the polygon to the circumference of the circle, we get

$$2nr\tan\frac{\pi}{n} = 2\pi r, \text{ when } n = \infty,$$

or

$$n\tan\frac{\pi}{n} = \pi, \text{ when } n = \infty. \qquad (1)$$

Then area of circle $=$ area of polygon when $n = \infty$,

$$\begin{aligned} &= n\tan\frac{\pi}{n}.r^2, \\ &= \pi r^2. \qquad (2) \end{aligned}$$

99. *To find the area of the sector and the segment of a circle.*

Let AOB be the sector, θ the circular measure of the vertical angle, r the radius of the circle.

Then

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\theta}{2\pi},$$

or

$$\text{area of sector} = \frac{\theta}{2\pi} \times \pi r^2 = \frac{r^2\theta}{2}. \qquad (1)$$

Area of segment $ADB =$ sector $AOB -$ triangle AOB, (Fig. Art. 96.)

$$\begin{aligned} &= \frac{r^2\theta}{2} - \frac{r^2\sin\theta}{2} \\ &= \frac{r^2}{2}(\theta - \sin\theta). \qquad (2) \end{aligned}$$

The student should examine this result when θ is greater than two right angles.

EXERCISE XIII.

1. The sides of a triangle are 6, 8 and 10 feet; find the radii of the inscribed, circumscribed and escribed circles.

2. Find the ratio between the radii of the inscribed and circumscribed circles of an equilateral triangle.

3. Find the angle subtended at the circumference of a circle 10 feet in diameter by a chord 5 feet in length.

4. The diagonals of a quadrilateral are 18 and 20 feet in length, and contain an angle of $37\frac{1}{2}^\circ$; find its area.

5. Find the length of the lines joining the centre of the inscribed circle to the angles of the triangle.

6. Find the distances between the centre of the inscribed and the centres of the escribed circles.

7. Find the lengths of the lines joining the centres of the escribed circles to the angles of the triangle, and thence the distances between the centres of the escribed circles.

8. Find the angles of the triangle formed by joining the centres of the escribed circles.

9. Find the area of the triangle formed by joining the centres of the escribed circles.

10. Show that the radius of the circle passing through the centres of the escribed circles is double the radius of the circle circumscribing the original triangle.

11. If a series of triangles of the same perimeter be described about the same circle they will have equal areas.

12. The two triangles in the ambiguous case may each be inscribed in the same circle.

13. If R be the radius of a circle circumscribing a triangle then $R = \dfrac{b - a\cos C}{2\cos A \sin C}$.

14. Show geometrically, and also symbolically, that the area of a triangle is $s(s-a)\tan\dfrac{A}{2}$.

15. Show that the perpendicular from an angle of a triangle upon the opposite side is an harmonic mean between the radii of the escribed circles opposite the remaining angles.

16. Prove geometrically

$$(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2},$$

and thence deduce $r=s\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$.

17. Express a side of a triangle in terms of the radius of the inscribed circle and the adjacent half angles of the triangle, and then deduce

$$r=\frac{a}{\cot\frac{B}{2}+\cot\frac{C}{2}}=\frac{b}{\cot\frac{C}{2}+\cot\frac{A}{2}}=\frac{c}{\cot\frac{A}{2}+\cot\frac{B}{2}}.$$

18. From the preceding example deduce

$$r=\frac{s}{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}=\frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}.$$

19. In Art. 93, show that $BF=s-c$, $EC=s-b$, and thence prove $r_1=(s-c)\cot\frac{B}{2}=(s-b)\cot\frac{C}{2}$.

20. Prove symbolically that $r_1\left(\cot\frac{A}{2}-\tan\frac{B}{2}\right)=c$, and also verify the equality from the figure of Art. 93.

21. Prove $r_1=\dfrac{a}{\tan\frac{B}{2}+\tan\frac{C}{2}}=\dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$.

22. Prove $\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}=\dfrac{1}{r}$, and $r_1r_2+r_2r_3+r_3r_1=s^2$.

23. From two different expressions for the radius of the inscribed circle, each obtained geometrically, deduce the formula for $\tan\frac{A}{2}$ in terms of the sides.

24. From the value of $\tan \frac{A}{2}$ obtained as in the preceding example, deduce algebraically the values of $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$.

25. From two expressions for the area of a triangle obtained geometrically, deduce the value of $\sin A$ in terms of the sides, and thence prove $\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$.

26. From the value of $\sin A$ deduce that of $\cos A$, and then prove $\cos A = 2 \cos^2 \frac{A}{2} - 1 = 1 - 2 \sin^2 \frac{A}{2}$.

27. Assuming that the values of $\cos \frac{A}{2}$ and $\sin \frac{A}{2}$ are *real*, and that the value of $\cos A$ is numerically less than unity, prove Euc. I., 20, from each of these expressions separately.

28. From $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, and the fact that $\sin A + \sin B > \sin (A + B)$ obtained geometrically, prove Euc. I., 20; also prove the converse.

29. From $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, and $A + B + C = 180°$, deduce the values of $\sin A$ and $\cos A$; also prove $c = a \cos B + b \cos A$.

30. From $c = a \cos B + b \cos A$, and two similar equations, prove Euc. II., 12 and 13.

31. Deduce the values of the ratios for the half angles from two different expressions for the radius of an escribed circle.

32. If r be the radius of the inscribed circle, and r_a the radius of the circle inscribed between this circle and the sides containing the angle A, then

$$r_a = r \cdot \frac{1 - \sin \frac{A}{2}}{1 + \sin \frac{A}{2}}.$$

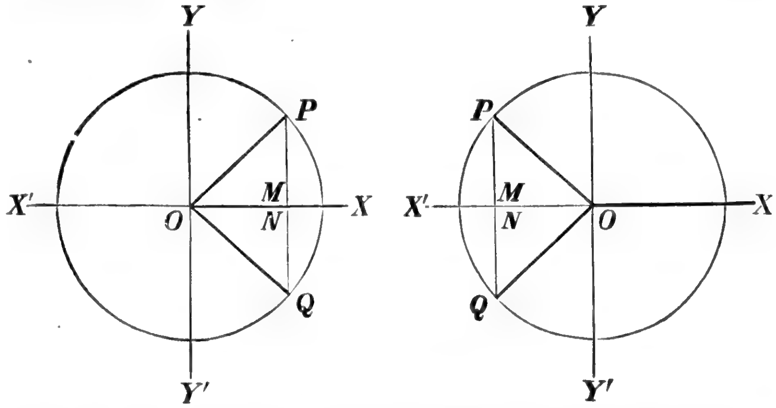
CHAPTER VII.

RATIOS FOR RELATED ANGLES.

100. The connection between the ratios for complementary and supplementary angles has already been pointed out for such angles as occur in connection with triangles. We now proceed to generalize the results already obtained, and to investigate other formulæ of a similar character.

In this and the following chapters we are especially careful to observe *directions* as well as *magnitude.*

101. *To compare the ratios of two angles equal in magnitude, but described in opposite directions.*



Take two straight lines, $X'OX$, $Y'OY$, at right angles to each other as axes of references. Let equal lines OP, OQ revolve through equal angles in opposite directions, starting from OX. Denote XOP by A, then XOQ is $-A$. Draw PM, QN perpendicular to XX'. Then triangles POM, QON are geometrically equal. Euc. I., 26.

And since P and Q are always on opposite sides of XX', but on the same side of YY', we have

$$NQ = -MP, \text{ and } ON = OM$$

for all values of the angles involved.

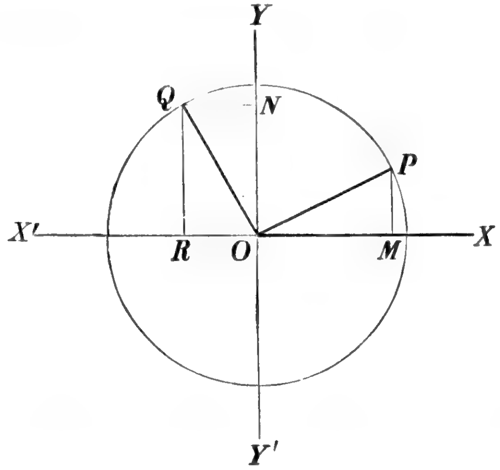
Then,

$$\sin(-A) = \sin XOQ = \frac{NQ}{OQ} = \frac{-MP}{OP} = -\sin XOP = -\sin A,$$

$$\cos(-A) = \cos XOQ = \frac{ON}{OQ} = \frac{OM}{OP} = \cos XOP = \cos A.$$

Similarly prove $\tan(-A) = -\tan A$, $\sec(-A) = \sec A$,
$\cot(-A) = -\cot A$, $\operatorname{cosec}(-A) = -\operatorname{cosec} A$.

102. *To compare the trigonometrical ratios of $90° + A$ with those of A.*



Let equal lines OP, OQ revolve through equal angles in the same direction, starting from OX, OY respectively. Denote XOP by A, then XOQ is $90° + A$. Draw PM, QR perpendicular to XX', and QN perpendicular to YY'. Then triangles POM, QON, are geometrically equal (Euc. I., 26). Now when P is above XX', Q is to the left of YY', and consequently MP

and NQ have contrary signs; and when P is to the right of YY', Q is above XX', and consequently OM and RQ have the same sign.

Therefore $\quad RQ = ON = OM$, and $NQ = OR = -MP$.

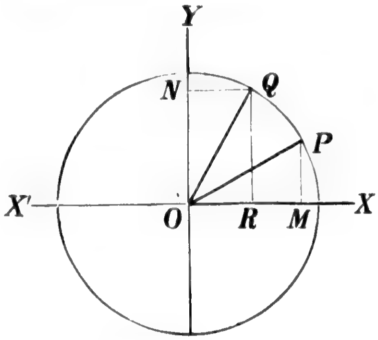
Then

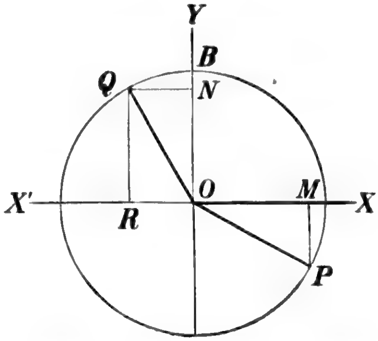
$$\sin(90^\circ + A) = \sin XOQ = \frac{RQ}{OQ} = \frac{OM}{OP} = \cos XOP = \cos A,$$

$$\cos(90^\circ + A) = \cos XOQ = \frac{OR}{OQ} = \frac{-MP}{OP} = -\sin XOP = -\sin A.$$

Similarly prove $\tan(90^\circ + A) = -\cot A$, $\sec(90^\circ + A) = -\operatorname{cosec} A$
$\cot(90^\circ + A) = -\tan A$, $\operatorname{cosec}(90^\circ + A) = \sec A$

103. *To compare the trigonometrical ratios of any angle with those of its complement.*





Let equal lines OP, OQ revolve through equal angles in opposite directions, starting from OX, OY respectively. Denote XOP by A, then XOQ is $90^\circ - A$, and these angles are complementary. Draw PM, QR perpendicular to OX, and QN perpendicular to OY. Then triangles POM, QON are geometrically equal (Euc. I., 26). Now when P is above XX', Q is to the right of YY', therefore MP and NQ have the same sign; and when P is to the right of YY', Q is above XX', therefore OM and RQ have the same sign.

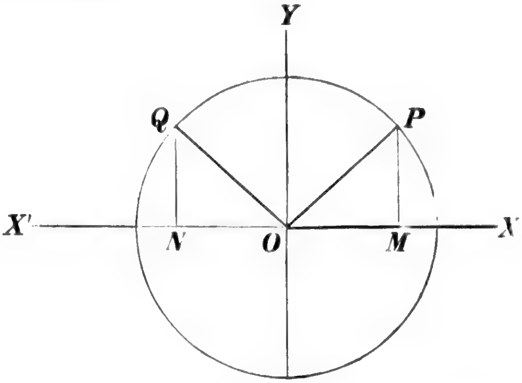
Therefore $\quad MP = NQ = OR$, and $OM = ON = RQ$.

Then $\sin(90° - A) = \sin XOQ = \dfrac{RQ}{OQ} = \dfrac{OM}{OP} = \cos XOP = \cos A$

$$\cos(90° - A) = \cos XOQ = \frac{OR}{OQ} = \frac{MP}{OP} = \sin XOP = \sin A.$$

Similarly prove $\tan(90° - A) = \cot A, \quad \sec(90° - A) = \operatorname{cosec} A,$
$\cot(90° - A) = \tan A, \operatorname{cosec}(90° - A) = \sec A.$

104. *To compare the trigonometrical ratios of any angle with those of its supplement.*



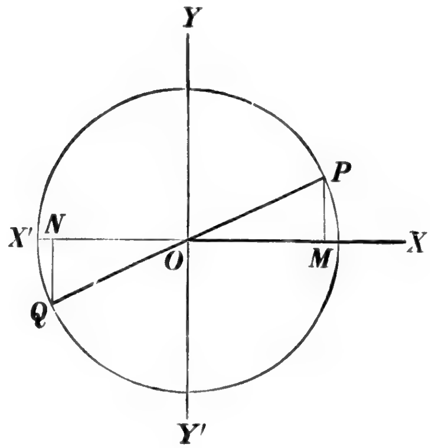
Let equal straight lines OP, OQ revolve through equal angles in opposite directions, starting from OX, OX' respectively. Denote the angle XOP by A, then XOQ is $180° - A$, and these angles are supplementary. Draw PM, QN perpendicular to XOX', then triangles POM, QON are geometrically equal (Euc. I., 26). Now P and Q are always on the same side of XX', but on opposite sides of YY', therefore we have $MP = NQ$ and $ON = -OM$ for all values of the angles involved. Then

$$\sin(180° - A) = \sin XOQ = \frac{NQ}{OQ} = \frac{MP}{OP} = \sin XOP = \sin A,$$

$$\cos(180° - A) = \cos XOQ = \frac{ON}{OQ} = \frac{-OM}{OP} = -\cos XOP = -\cos A.$$

Similarly prove $\tan(180° - A) = -\tan A, \quad \cot(180° - A) = -\cot A$
$\sec(180° - A) = -\sec A, \operatorname{cosec}(180° - A) = \operatorname{cosec} A.$

105. *To compare the trigonometrical ratios of $180^\circ + A$ with those of A.*



Let equal straight lines OP, OQ revolve through equal angles in the same direction, starting from OX, OX' respectively. Denote XOP by A, then XOQ is $180^\circ + A$. Draw MP, QN perpendicular to XX', then triangles POM, QON, are geometrically equal (Euc. I., 26). Now since P and Q are always on opposite sides both of XX' and YY', we have

$$NQ = -MP, \text{ and } ON = -OM$$

for all values of the angles involved.

Therefore

$$\sin(180^\circ + A) = \sin XOQ = \frac{NQ}{OQ} = \frac{-MP}{OP} = -\sin XOP = -\sin A,$$

$$\cos(180^\circ + A) = \cos XOQ = \frac{ON}{OQ} = \frac{-OM}{OP} = -\cos XOP = -\cos A.$$

Similarly prove

$$\tan(180^\circ + A) = \tan A. \qquad \sec(180^\circ + A) = -\sec A.$$
$$\cot(180^\circ + A) = \cot A. \qquad \operatorname{cosec}(180^\circ + A) = -\operatorname{cosec} A.$$

EXERCISE XIV.

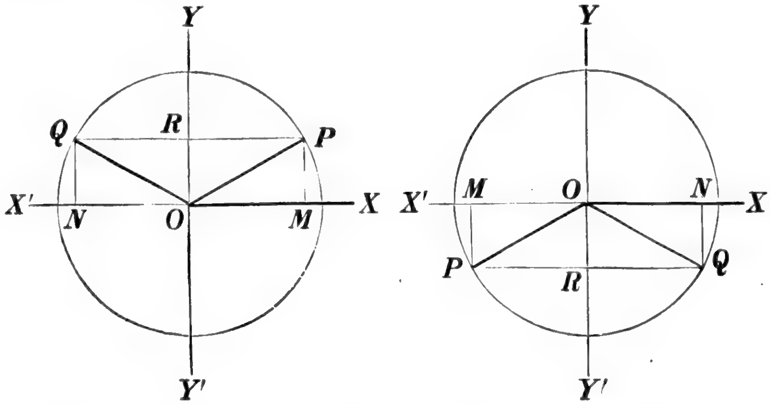
Prove the following (1) geometrically, (2) by the aid of preceding formulæ :

1. $\sin 100° = \cos 10°$.
2. $\tan 100° = -\cot 10°$.
3. $\cos 100° = -\sin 10°$.
4. $\cos 300° = \sin 30°$.
5. $\tan 225° = \tan 45°$.
6. $\sin 225° = \cos 135°$.
7. $\sin 240° = \sin(-120°) = -\cos 30°$.
8. $\cos 175° = \sin 265° = -\cos 5°$.
9. $\sec 700° = \sec 20° = \operatorname{cosec} 110°$.
10. $\cot 330° = \tan(-60°) = -\cot 30°$.
11. $\tan\left(5\pi + \frac{2\pi}{3}\right) = -\cot\frac{\pi}{6}$.
12. $\operatorname{cosec}\left(7\pi + \frac{\pi}{4}\right) = -\sec\frac{\pi}{4}$.

If A, B, C be the angles of a triangle, prove the following :

13. $\sin A = \sin(B+C)$, $\cos B = -\cos(C+A)$, $\tan C = -\tan(A+B)$.
14. $\sin\frac{A}{2} = \cos\frac{B+C}{2}$, $\cos\frac{B}{2} = \sin\frac{C+A}{2}$, $\tan\frac{C}{2} = \cot\frac{A+B}{2}$.
15. $\sin(2A+B+C) = -\sin A$, $\tan(A+2B+C) = \tan B$.
16. $\sin(A+B-C) = \sin 2C$, $\cos(A-B+C) = -\cos 2C$.
17. $\tan(A-B-C) = \tan 2A$, $\cot(A-B+C) = -\cot 2B$.
18. $\sec\frac{B-C}{2} = \operatorname{cosec}\frac{A+2B}{2}$, $\tan\frac{C-A}{2} = -\cot\frac{B+2C}{2}$.
19. Given $\sin(90° - A) = \cos A$, prove $\cos(90° - A) = \sin A$.
20. From the ratios of $90° + A$ deduce those of $180° + A$.
21. From the ratios of $90° + A$ and $-A$ derive those of $90° - A$, $180° - A$, and $180° + A$.
22. Given the ratios of $90° + A$ and $90° - A$, derive those of $180° - A$, $180° + A$, and $-A$.
23. Draw two angles, A and B, such that $\sin A + \sin B = 0$, and $\cos A + \cos B = 0$.

106. *To find all the angles which have a given sine,* i.e., *to solve the equation,* $\sin\theta = a$.



Describe a circle of unit radius and draw the diameters XOX', YOY' at right angles to each other. Let OR, measured on OY (or OY'), represent a in magnitude and sign; through R draw PQ parallel to XX', meeting the circle in P and Q; join OP, OQ; then any angle of which OX is the initial, and either OP or OQ the final line, but no other angle, will have its sine equal to a.

Denote XOP by α, then XOQ is $\pi - \alpha$.

As the revolving line passes the points P and Q in successive revolutions in the positive direction we obtain the series of angles,

$$\alpha,\ \pi-\alpha,\ 2\pi+\alpha,\ 3\pi-\alpha,\ 4\pi+\alpha,\ 5\pi-\alpha\ldots\text{etc.}$$

Similarly from the negative direction we get,

$$-\pi-\alpha,\ -2\pi+\alpha,\ -3\pi-\alpha,\ -4\pi+\alpha,\ -5\pi-\alpha\ldots\text{etc.}$$

Now observe that

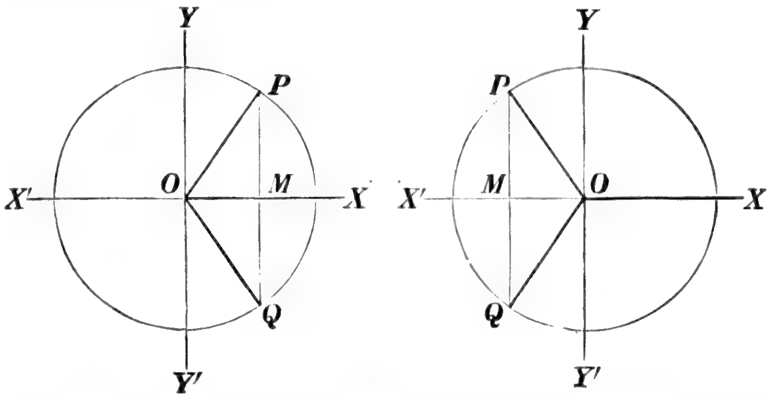
1. These series contain all integral multiples of π, both positive and negative.

2. To each even multiple of π, α is added, but from each odd multiple, α is subtracted.

Therefore $\theta = n\pi + (-1)^n\overset{a}{\alpha}$, in which n denotes any integer, positive or negative, gives without excess or defect the series of angles required.

The same formula gives all the angles whose cosecant is the reciprocal of the given sine.

107. *To find all the angles which have a given cosine,* i.e., *to solve the equation* $\cos\theta = a$.



Describe a circle of unit radius and draw the diameters XOX', YOY', at right angles to each other. Let OM, measured on OX (or OX'), represent a in magnitude and sign; through M draw PMQ parallel to YY', meeting the circle in P and Q; join OP, OQ; then any angle described from OX to either OP or OQ, but no other angle, will have its cosine equal to a.

Denote XOP by α, then XOQ is $-\alpha$.

As the revolving line passes the points P and Q in successive revolutions in the positive direction, we obtain the series of angles

$$\alpha,\ 2\pi-\alpha,\ 2\pi+\alpha,\ 4\pi-\alpha,\ 4\pi+\alpha, \ldots \text{etc.}$$

Similarly from the negative direction we get,

$$-\alpha,\ -2\pi+\alpha,\ -2\pi-\alpha,\ -4\pi+\alpha,\ \overset{4}{-}\pi-\alpha \ldots \text{etc.}$$

Now observe that,

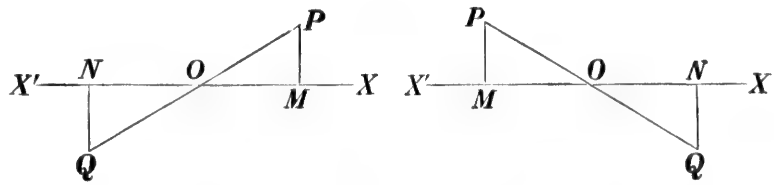
1. These series contain only the even multiples of π, positive or negative.

2. To each of these multiples either $+\alpha$ or $-\alpha$ may be added.

Therefore $\theta = 2n\pi \pm \alpha$, in which n denotes any integer, positive or negative, gives without excess or defect the series of angles required.

The same formula gives all the angles whose secant is the reciprocal of the given cosine.

108. *To find all the angles which have a given tangent*, i.e., *to solve the equation* $\tan\theta = a$.



From O in the horizontal line $X'X$ draw OM a unit in length in the direction indicated by the sign of a. Draw MP at right angles to $X'X$ in the positive direction and of the length indicated by a; produce PO to Q, making OQ equal OP, and draw QN perpendicular to $X'X$; then any angle described from OX to either OP or OQ will have its tangent equal to a.

Denote XOP by α, then XOQ is $\pi + \alpha$.

As the revolving line passes the points P and Q in the positive direction we get the series of angles,

$$\alpha,\ \pi + \alpha,\ 2\pi + \alpha,\ 3\pi + \alpha,\ 4\pi + \alpha, \text{ etc.}$$

Similarly from the negative direction we get,

$$-\pi + \alpha,\ -2\pi + \alpha,\ -3\pi + \alpha,\ -4\pi + \alpha, \text{ etc.}$$

Now observe that

1. These series contain all the multiples of π, both positive and negative.

2. To each multiple α is added.

Therefore $\theta = n\pi + \alpha$, in which n denotes any integer, positive or negative, gives without excess or defect the series of angles required.

The same formula gives all the angles whose cotangent is the reciprocal of the given tangent.

109. In the formulæ of Arts. 106-108, α represents the smallest positive angle whose ratio has the required value. This restriction, however, is not necessary. The angle α may be replaced by any angle having the required ratio, and the formulæ will still be true. In Art. 106 let β be any angle such that $\sin\theta = \sin\beta = a$.

Then $\beta = r\pi + (-1)^r\alpha$, in which r denotes some integer.

Therefore,
$$\begin{aligned} n\pi + (-1)^n\beta &= n\pi + (-1)^n r\pi + (-1)^{n+r}\alpha \\ &= \{n + (-1)^n r\}\pi + (-1)^{n+r}\alpha \\ &= m\pi + (-1)^m\alpha, \end{aligned}$$

in which $m = n \pm r$, according as n is even or odd. In either case m represents an integer and may consequently be replaced by n. The series of angles represented by $n\pi + (-1)^n\alpha$ and $n\pi + (-1)^n\beta$ are thus identical. Similarly the other formulæ may be shown to be universally true.

110. Since every numerical quantity has two square roots, we see from Art. 54, that for any given value of the sine of an angle there are *two* corresponding values for the cosine, tangent, secant and cotangent, and the geometrical meaning of this double sign may now be explained. In the diagram of Art. 106, $OR = a$, and $OP = OQ = 1$.

Then, from the figure on the left hand, we have

$$OM = +\sqrt{1-a^2}, \text{ and } ON = -\sqrt{1-a^2},$$

with the signs reversed for the figure on the right.

So that when $\sin\theta = a$ we may have either

$$\cos\theta = \frac{OM}{OP} = +\sqrt{1-a^2}, \text{ or } \cos\theta = \frac{ON}{OQ} = -\sqrt{1-a^2},$$

according as we take XOP, or XOQ, as the value of θ. Both these results are included in the single algebraical formula,

$$\cos\theta = \sqrt{1-\sin^2\theta}.$$

Similarly it may be shown that when the value of any one ratio is given there are two corresponding values for all the remaining ratios excepting the reciprocal of the given ratio.

EXERCISE XV.

Obtain complete solutions of the following equations:

1. $\sin\theta = 1$. 2. $\cos\theta = 1$. 3. $\tan\theta = 1$.

4. $\tan\theta = -1$. 5. $\cot\theta = \sqrt{3}$. 6. $\operatorname{cosec}\theta = 2$.

7. $\sec\theta = \dfrac{2}{\sqrt{3}}$. 8. $\sin^2\theta = \frac{1}{4}$. 9. $\cos^2\theta = \frac{1}{2}$.

10. $\sin\theta = \cos\theta$. 11. $\sin^2\theta = \cos^2\theta$. 12. $\tan^2\theta = \tan^2\alpha$.

13. $\sin 2\theta = \frac{1}{2}$. 14. $2\sin A = \tan A$. 15. $\tan p\theta = \cot q\theta$.

16. $\tan\theta + \cot\theta = 1$. 17. $\sec^3\theta - 2\tan^2\theta = 2$.

18. $\sin\theta + \operatorname{cosec}\theta = \frac{5}{2}$. 19. $\sin 5x + \sin 3x = \cos x$.

20. $\cot\theta - \tan\theta = \cos\theta + \sin\theta$. 21. $\sin 9\theta + \sin 5\theta + 2\sin^2\theta = 1$.

22. $\sin(A+B) = \dfrac{\sqrt{3}}{2}$, $\tan(A-B) = 1$.

23. Prove that if a series of angles have a common tangent they are in arithmetical progression. Is the converse true?

24. If two angles have the same Sine ~~sign~~, show that either their sum is an odd multiple, or their difference is an even multiple of π. State a similar theorem with regard to the cosine.

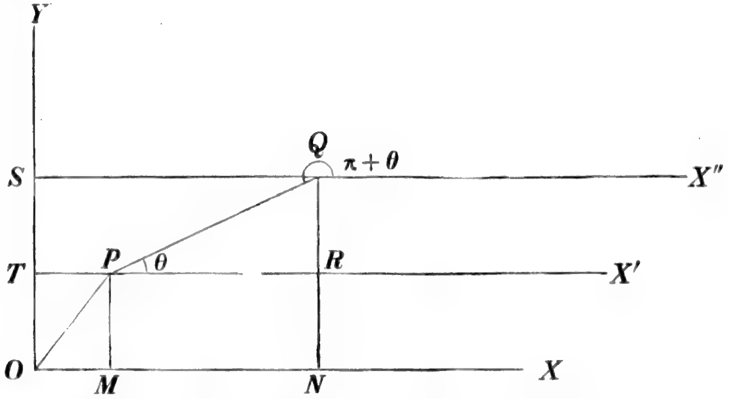
25. Find all the angles which have both their sines and their cosines equal.

26. The circumference of a carriage wheel is 15 feet, and the carriage is moving forward at the rate of 10 miles per hour; at what intervals of time will a point in the wheel which at first rested on the ground be at a height of $\frac{1}{4}$ the diameter?

CHAPTER VIII.

RATIOS OF COMPOUND ANGLES.

111. The angle which one trigonometrical line, PQ, makes with another, OX, is thus estimated. Move the line OX parallel to itself until its initial point O coincides with the initial point P of PQ, and let PX' be the line so placed; then the angle described by a line revolving from PX' to PQ is the angle required.



Similarly the angle $X''QP$ is the angle which the line QP makes with OX, and if PQ makes with OX the angle θ, it is evident that QP makes with the same line the angle $\pi+\theta$.

112. *If* PQ *make with* OX *an angle* θ, *the projection of* PQ *on* OX *is* PQ $\cos\theta$; *and if* OY *make with* OX *a positive right angle, the projection of* PQ *on* OY *is* PQ $\sin\theta$.

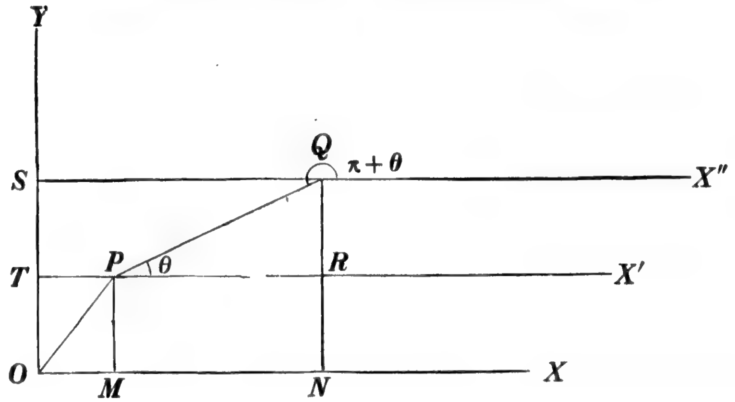
Through P and Q draw RPT, QS, parallel to OX; PM, and QRN perpendicular to OX; then $\angle RPQ = \theta$, MN = projection of PQ on OX, TS = projection on OY. Art. 7.

Now by definition, Art. 43, we have

$$\frac{PR}{PQ} = \cos\theta, \quad \frac{RQ}{PQ} = \sin\theta.$$

Therefore, for all values of θ,

$$MN = PR = PQ\cos\theta, \quad TS = RQ = PQ\sin\theta.$$



113. One point in the preceding article requires careful attention. When we speak of the angle θ at the point P we tacitly assume the direction from P to Q to be positive, but when we speak of the angle $\pi + \theta$ at Q, we assume the positive direction to be from Q to P. Clearly, then, we must not assume QP in the latter case to be the negative of PQ in the former; each is positive in connection with its own angle. If we denote the length of PQ by l, we have

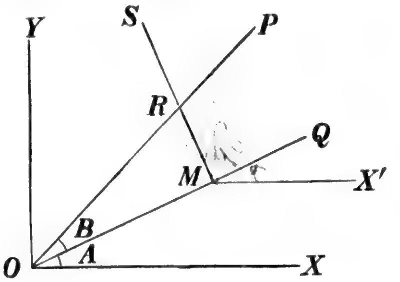
$$\text{proj. of } PQ = l\cos\theta, \quad \text{proj. of } QP = l\cos(\pi + \theta) = -l\cos\theta,$$

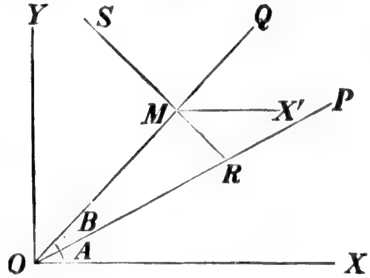
which shows the projection of QP to be the negative of that of PQ, as it should be.

If, however, we choose to consider the line QP to be projected

to be negative and denote it by $-l$, then PQ is positive and θ is the angle to be chosen. When, for any reason, a line has already been considered negative, this is the preferable mode of proceeding.

114. *To find the sine and the cosine of the sum and the difference of two angles in terms of the sines and cosines of the angles.*





Let the line OP starting from OX trace out successively the angles $XOQ = A$, and $QOP = B$, then OP makes with OX the angle $A + B$. From M, any point in OQ, draw MS, making a positive right angle with OQ and meeting OP in R, either line being produced if necessary; then MS makes with OX the angle $A + 90°$.

Now $\quad OM = OR \cos B, \quad MR = OR \sin B,$ Art. 112.

$\quad \cos(A + 90°) = -\sin A, \quad \sin(A + 90°) = \cos A.$ Art. 102.

Since projection of OR on OX equals sum of projections of OM and MR on OX we have Art. 10.

$$OR \cos(A + B) = OM \cos A + MR \cos(A + 90°)$$
$$= OR \cos B \cos A - OR \sin B \sin A.$$

Therefore $\quad \cos(A + B) = \cos A \cos B - \sin A \sin B.$ (1)

Similarly projecting OR on OY, we have

$$OR \sin(A + B) = OM \sin A + MR \sin(A + 90)$$
$$= OR \cos B \sin A + OR \sin B \cos A.$$

Therefore $\quad \sin(A + B) = \sin A \cos B + \cos A \sin B.$ (2)

Equations (1) and (2) being proved for all values of A and B, change B into $-B$.

Then $\cos\{A+(-B)\} = \cos A \cos(-B) - \sin A \sin(-B)$,

or $\cos(A-B) = \cos A \cos B + \sin A \sin B$. (3)

And $\sin\{A+(-B)\} = \sin A \overset{\cos}{\cancel{\sin}}(-B) + \cos A \sin(-B)$,

or, $\sin(A-B) = \sin A \cos B - \cos A \sin B$. (4)

115. The proof given in the preceding article is perfectly general, and Fig. 1, which represents both angles as positive, is alone necessary. It will, however, be a valuable exercise to adapt this proof to Fig. 2, which represents the second angle as negative. Note then the following points:

1. OQ is positive, being a bounding line of angle A.

2. MS is positive, because it makes a positive right angle with OQ.

3. MR is negative, because it is drawn in a direction opposite to MS.

4. $X'MS = A + 90°$ is the angle which must be chosen to project the negative line MR on OX. Art. 113.

Then $OR \cos(A-B) = OM \cos A - MR \cos(A+90)$

$= OR \cos B \cos A - OR \sin B(-\sin A)$,

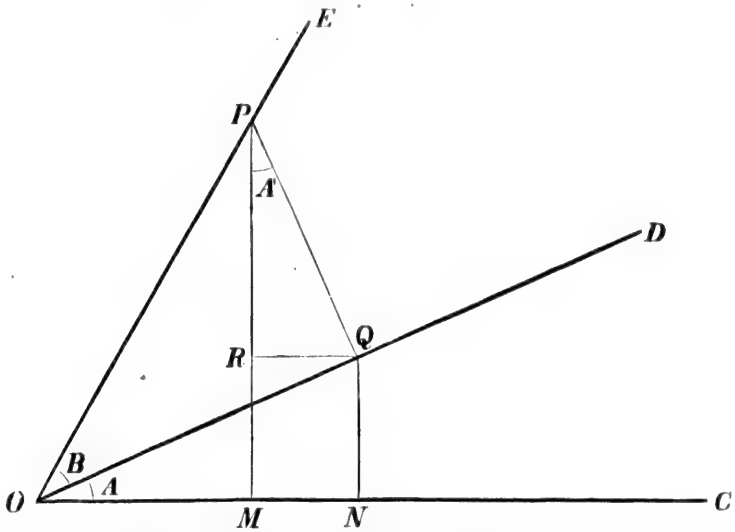
or $\cos(A-B) = \cos A \cos B + \sin A \sin B$.

As an exercise the student should draw the figure for various values of the angles, positive and negative, and adapt the given proof to each.

116. The very important formulæ of the preceding article may be proved in many different ways. We give another method, somewhat simpler than the former but less valuable, inasmuch as it is valid only for positive angles, such that $A+B < 90°$ and $A > B$.

117. *To prove*

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$



Let the angle COD be denoted by A, and DOE by B; then the angle COE will be denoted by $A+B$. In OE take any point P, draw PM and PQ perpendicular to OC and OD; draw QN and QR perpendicular to OC and PM.

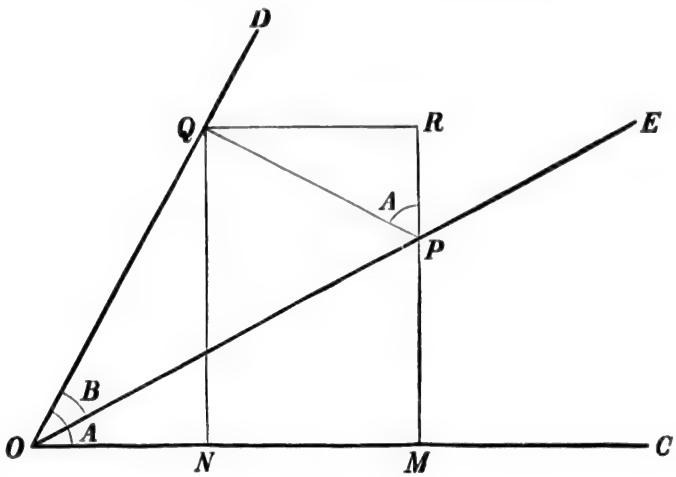
Then $$\angle QPR = 90° - PQR = RQO = A.$$

Now $$\sin(A+B) = \frac{MP}{OP} = \frac{MR+RP}{OP} = \frac{NQ}{OP} + \frac{RP}{OP}$$
$$= \frac{NQ}{OQ}\cdot\frac{OQ}{OP} + \frac{RP}{PQ}\cdot\frac{PQ}{OP}$$
$$= \sin A \cos B + \cos A \sin B. \qquad (1)$$

And $$\cos(A+B) = \frac{OM}{OP} = \frac{ON - MN}{OP} = \frac{ON}{OP} - \frac{RQ}{OP}$$
$$= \frac{ON}{OQ}\cdot\frac{OQ}{OP} - \frac{RQ}{PQ}\cdot\frac{PQ}{OP}$$
$$= \cos A \cos B - \sin A \sin B. \qquad (2)$$

118. *To prove*

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$



Denote the angle COD by A and DOE by B; then the angle COE will be denoted by $A-B$. In OE take any point P, draw PM and PQ perpendicular to OC and OD; draw QN and QR perpendicular to OC and MP produced.

Then $$\angle QPR = 90° - PQR = RQD = A.$$

Now $$\sin(A-B) = \frac{MP}{OP} = \frac{MR-RP}{OP} = \frac{NQ}{OP} - \frac{RP}{OP}$$
$$= \frac{NQ}{OQ} \cdot \frac{OQ}{OP} - \frac{RP}{QP} \cdot \frac{QP}{OP}$$
$$= \sin A \cos B - \cos A \sin B. \qquad (1)$$

And $$\cos(A-B) = \frac{OM}{OP} = \frac{ON+NM}{OP} = \frac{ON}{OP} + \frac{QR}{OP}$$
$$= \frac{ON}{OQ} \cdot \frac{OQ}{OP} + \frac{QR}{QP} \cdot \frac{QP}{OP}$$
$$= \cos A \cos B + \sin A \sin B. \qquad (2)$$

119. From the fundamental formulæ of Art. 114, many important results are easily obtained. We give a few examples.

Ex. 1.—Tan $(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)} = \dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

the last step being obtained by dividing both numerator and denominator of the preceding fraction by $\cos A \cos B$.

Ex. 2.—Sin $2A = \sin(A+A) = \sin A \cos A + \cos A \sin A$

$$= 2 \sin A \cos A.$$

Ex. 3.—Sin $(A+B+C) = \sin(A+B)\cos C + \cos(A+B)\sin C$

$= (\sin A \cos B + \cos A \sin B)\cos C + (\cos A \cos B - \sin A \sin B)\sin C$

$= \sin A \cos B \cos C + \sin B \cos C \cos A + \sin C \cos A \cos B -$
$\sin A \sin B \sin C.$

EXERCISE XVI.

1. Given the sines and cosines of 45° and 30°, prove

$$\sin 75° = \frac{\sqrt{3}+1}{2\sqrt{2}},\ \cos 75° = \frac{\sqrt{3}-1}{2\sqrt{2}},\ \tan 15° = 2 - \sqrt{3}.$$

2. If $\sin A = \cos B = \frac{3}{5}$, find $\sin(A+B)$ and $\cos(A+B)$.

3. If $\sin A = \frac{3}{5}$, and $\sin B = \frac{5}{13}$, find $\sin(A+B)$ and $\cos(A-B)$ (1) A and B both acute, (2) A acute and B obtuse.

4. If $\sin A = \dfrac{1}{\sqrt{5}}$, and $\sin B = \dfrac{1}{\sqrt{10}}$, and both angles lie in the first quadrant, then $A + B = 45°$. Find $A + B$ if each angle lies in the second quadrant.

5. Prove $\cos A + \sin A = \sqrt{2}\sin(A+45) = \sqrt{2}\cos(A-45°)$.

6. Prove $\cos A - \sin A = \sqrt{2}\sin(45° - A) = \sqrt{2}\cos(45° + A)$.

7. If $\tan A = \frac{1}{2}$, and $\tan B = \frac{1}{3}$, find $\tan(A+B)$.

8. If $\tan A = \frac{1}{70}$, $\tan B = \frac{1}{99}$, find $\tan(A-B)$.

9. If $\tan A = \frac{1}{5}$, find $\tan 4A$, and thus show that A is slightly greater than $11\frac{1}{4}°$.

10. If $\tan 2A = 2 \tan (A + B)$ then $\tan B = \tan^3 A$.

11. Prove $\sin (A + B) \cos B - \cos (A + B) \sin B = \sin A$.

12. Prove $\dfrac{\tan (\alpha - \beta) + \tan \beta}{1 - \tan (\alpha - \beta) \tan \beta} = \tan \alpha$.

13. Write down other identities employing the principle exemplified in examples 11 and 12.

14. If $\tan \theta = 2m + 1$ and $\cot (\theta - \phi) = 2m^2$, find $\tan \phi$.

EXERCISE XVII.

Prove the following identities and memorize Nos. 1-8.

1. $\sin (A + B) + \sin (A - B) = 2 \sin A \cos B$.

2. $\sin (A - B) - \sin (A - B) = 2 \cos A \sin B$.

3. $\cos (A - B) + \cos (A + B) = 2 \cos A \cos B$.

4. $\cos (A - B) - \cos (A + B) = 2 \sin A \sin B$.

5. $\sin (A + B) \sin (A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$.

6. $\cos (A + B) \cos (A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$.

7. $\tan (A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

8. $\tan (A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$.

9. $\tan (A + 45^\circ) = \dfrac{1 + \tan A}{1 - \tan A}$. 10. $\tan(A - 45) = \dfrac{\tan A - 1}{\tan A + 1}$.

11. $\cot (A + B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$.

12. $\cot (A - B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$.

13. $\tan A + \tan B = \dfrac{\sin (A + B)}{\cos A \cos B}$. 14. $\tan A - \tan B = \dfrac{\sin (A - B)}{\cos A \cos B}$.

15. $\cot A + \tan A = 2 \operatorname{cosec} 2A$. 16. $\cot A - \tan A = 2 \cot 2A$.

17. $\cot A + \tan B = \dfrac{\cos (A - B)}{\sin A \cos B}$. 18. $\cot A - \tan B = \dfrac{\cos (A + B)}{\sin A \cos B}$.

19. $\dfrac{\cot \theta + \tan \phi}{\cot \theta - \tan \phi} = \dfrac{\tan \theta \tan \phi + 1}{1 - \tan \theta \tan \phi} = \dfrac{1 + \cot \theta \cot \phi}{\cot \theta \cot \phi - 1} = \dfrac{\cos (\theta - \phi)}{\cos (\theta + \phi)}$.

20. $\dfrac{\tan\theta+\tan\phi}{\tan\theta-\tan\phi}=\dfrac{\tan\theta\cot\phi+1}{\tan\theta\cot\phi-1}=\dfrac{1+\cot\theta\tan\phi}{1-\cot\theta\tan\phi}=\dfrac{\sin(\theta+\phi)}{\sin(\theta-\phi)}$.

21. $\sin(A+B+C)=\sin A\cos B\cos C+\sin B\cos C\cos A+\sin C\cos A\cos B-\sin A\sin B\sin C$.

22. $\cos(A+B+C)=\cos A\cos B\cos C-\sin A\sin B\cos C-\sin B\sin C\cos A-\sin C\sin A\cos B$.

23. $\tan(A+B+C)=\dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$.

120. For convenience of reference the important formulæ of Art. 114 are here repeated.

$$\sin(A+B)=\sin A\cos B+\cos A\sin B. \qquad (1)$$
$$\sin(A-B)=\sin A\cos B-\cos A\sin B. \qquad (2)$$
$$\cos(A+B)=\cos A\cos B-\sin A\sin B. \qquad (3)$$
$$\cos(A-B)=\cos A\cos B+\sin A\sin B. \qquad (4)$$

From these, by addition and subtraction we easily obtain

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B. \qquad (5)$$
$$\sin(A+B)-\sin(A-B)=2\cos A\sin B. \qquad (6)$$
$$\cos(A-B)+\cos(A+B)=2\cos A\cos B. \qquad (7)$$
$$\cos(A-B)-\cos(A+B)=2\sin A\sin B. \qquad (8)$$

Now let $A+B=P$ and $A-B=Q$,

then $$A=\frac{P+Q}{2}, \text{ and } B=\frac{P-Q}{2}.$$

Substituting these values for A and B we get

$$\sin P+\sin Q=2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}. \qquad (9)$$
$$\sin P-\sin Q=2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}. \qquad (10)$$
$$\cos Q+\cos P=2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}. \qquad (11)$$
$$\cos Q-\cos P=2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}. \qquad (12)$$

Again, re-arranging formulæ (5)....(8) we have

$$\sin A \cos B = \tfrac{1}{2}\{\sin(A+B) + \sin(A-B)\}. \qquad (13)$$
$$\cos A \sin B = \tfrac{1}{2}\{\sin(A+B) - \sin(A-B)\}. \qquad (14)$$
$$\cos A \cos B = \tfrac{1}{2}\{\cos(A-B) + \cos(A+B)\}. \qquad (15)$$
$$\sin A \sin B = \tfrac{1}{2}\{\cos(A-B) - \cos(A+B)\}. \qquad (16)$$

121. The importance of the preceding formulæ is such that we repeat a portion of them in words, with a few observations, to assist the learner in committing them accurately to memory.

1. The sum of the sines of two angles is equal to twice the sine of half their sum multiplied by the cosine of half their difference.

2. The difference of the sines of two angles is equal to twice the cosine of half their sum multiplied by the sine of half their difference.

3. The sum of the cosines of two angles is equal to twice the cosine of half their sum multiplied by the cosine of half their difference.

4. The difference of the cosines of two angles is equal to twice the sine of half their sum multiplied by the sine of half their difference.

122. When the sine of a difference occurs care must be taken to observe the proper order in subtracting, and we observe :

1. The angles forming the difference have the same order on both sides of the equality when the difference of sines is involved as in (10), but have the reverse order on opposite sides as in (12), when the difference of cosines is taken.

2. When the product of a sine and a cosine is to be transformed into a sum, subtract the *cosine angle* from the other and *add* the sine of the difference thus found to the sine of the sum of the two angles. Equations (13) and (14) are thus different modes of expressing the same relations.

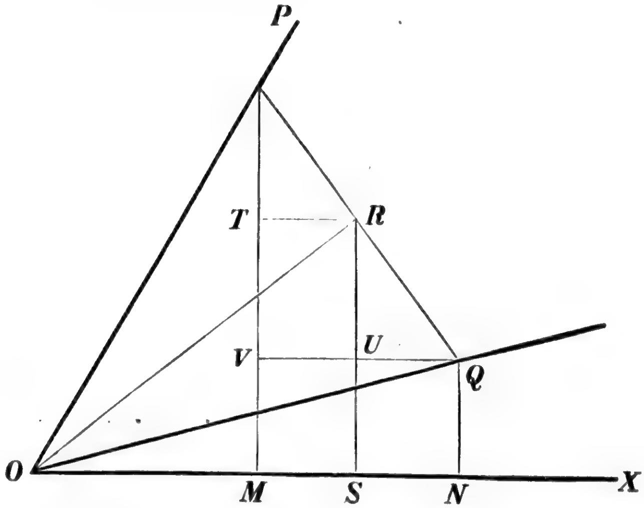
3. When the cosine of a difference occurs as in (9) and (11), either angle may be subtracted from the other. Art. 101.

4. In transforming the product of two sines as in (16), the *difference* precedes the *sum*.

123. It will be a valuable exercise for the student to prove equations (9)....(12) of Art. 120, geometrically. We give the proof of (9) as an example.

124. *To prove geometrically that*

$$\sin P + \sin Q = 2 \sin \frac{P+Q}{2} \cos \frac{P-Q}{2}.$$



Draw the angle $XOQ = Q$ and $XOP = P$; take OP and OQ each a unit in length; join PQ, bisect the angle QOP by OR which will consequently be at right angles to PQ; draw PM, RS, QN perpendicular to OX, and RT, QUV parallel to OX.

Then $$\text{angle } QOR = \frac{QOP}{2} = \frac{P-Q}{2},$$

and $$XOR = XOQ + QOR = Q + \frac{P-Q}{2} = \frac{P+Q}{2}.$$

Since $PR = RQ$ it easily follows that $PT = RU$,
and $TR = UQ$, and consequently $PM + QN = 2RS$.

Then
$$\sin P + \sin Q = PM + QN = 2RS$$
$$= 2OR \sin \frac{P+Q}{2}$$
$$= 2OP \cos \frac{P-Q}{2} \sin \frac{P+Q}{2}$$
$$= 2 \sin \frac{P+Q}{2} \cos \frac{P-Q}{2}.$$

Similarly equations (10), (11) and (12) may be proved. The fundamental formulæ (1)....(4) may also be proved from the same diagram.

EXERCISE XVIII.

Change the following sums and differences into products:

1. $\sin 75^\circ + \sin 15^\circ$.
2. $\sin 75^\circ - \sin 15^\circ$.
3. $\cos 60^\circ + \cos 30^\circ$.
4. $\cos 30^\circ - \cos 60^\circ$.
5. $\sin 75^\circ + \cos 30^\circ$.
6. $\cos 80^\circ - \sin 30^\circ$.
7. $\sin 3\theta + \sin \theta$.
8. $\cos 3\theta - \cos \theta$.
9. $\cos \frac{\pi}{3} + \cos \frac{2\pi}{3}$.
10. $\sin \frac{\pi}{3} + \cos \frac{\pi}{3}$.

Change the following products into a sum or a difference:

11. $2 \sin \theta \cos \phi$.
12. $2 \cos \alpha . \cos \beta$.
13. $\sin 2\theta \sin 2\phi$.
14. $\cos \theta \cos 3\theta$.
15. $\sin \alpha . \sin 5\alpha$.
16. $\sin \frac{5\theta}{2} . \sin \frac{3\theta}{2}$.
17. $\sin (A + B) \sin (A - B)$.
18. $\cos (A + B) \cos (A - B)$.
19. $\cos (30^\circ + A) \cos (150^\circ + A)$.
20. $4 \sin A \sin (120^\circ + A) \sin (240^\circ + A)$.

Prove the following identities:

21. $\dfrac{\sin 3^\circ + \sin 33^\circ}{\cos 3^\circ + \cos 33^\circ} = \tan 18^\circ$.
22. $\dfrac{\cos 3^\circ - \cos 33^\circ}{\sin 3^\circ + \sin 33^\circ} = \tan 15^\circ$.

23. $\dfrac{\cos 10^\circ + \sin 10^\circ}{\cos 10^\circ - \sin 10^\circ} = \tan 55^\circ$. 24. $\dfrac{\cos 25^\circ - \sin 5^\circ}{\cos 5^\circ - \sin 25^\circ} = \cot 35^\circ$.

25. $\sec 60^\circ - \sec 40^\circ = 4 \sin 10^\circ$. 26. $\cot 50^\circ + \tan 50^\circ = 2 \sec 10^\circ$.

27. $\sin \dfrac{\theta}{2} \sin \dfrac{7\theta}{2} + \sin \dfrac{3\theta}{2} \sin \dfrac{11\theta}{2} = \sin 2\theta \sin 5\theta$.

28. $\cos A + \cos (120^\circ + A) + \cos (120^\circ - A) = 0$.

29. $4 \sin A \sin (60^\circ + A) \sin (60^\circ - A) = \sin 3A$.

30. $4 \cos A \cos (60^\circ + A) \cos (60^\circ - A) = \cos 3A$.

31. $16 \cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = 1$.

32. $4 \sin 20^\circ \sin 40^\circ \sin 80^\circ = \sin 60^\circ$.

33. $\dfrac{\sin A - \sin B}{\cos A + \cos B} = \tan \dfrac{A - B}{2} = \dfrac{\cos B - \cos A}{\sin A + \sin B}$.

34. $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan \dfrac{A + B}{2} = \dfrac{\cos B - \cos A}{\sin A - \sin B}$.

35. $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}$.

36. $\dfrac{\cos A - \cos B}{\cos A + \cos B} = \tan \dfrac{A + B}{2} \tan \dfrac{B - A}{2} = \dfrac{\tan^2 \frac{1}{2} B - \tan^2 \frac{1}{2} A}{1 - \tan^2 \frac{1}{2} B \tan^2 \frac{1}{2} A}$.

37. $\tan \dfrac{\alpha + \beta}{2} + \tan \dfrac{\alpha - \beta}{2} = \dfrac{2 \sin \alpha}{\cos \alpha + \cos \beta}$.

38. $\cot \dfrac{\alpha + \beta}{2} + \cot \dfrac{\alpha - \beta}{2} = \dfrac{2 \sin \alpha}{\cos \beta - \cos \alpha}$.

39. $\dfrac{\sin A + \sin (A + B) + \sin (A + 2B)}{\cos A + \cos (A + B) + \cos (A + 2B)} = \tan (A + B)$.

40. $\dfrac{\sin A + 2 \sin 3A + \sin 5A}{\sin 3A + 2 \sin 5A + \sin 7A} = \dfrac{\sin 3A}{\sin 5A}$.

41. If $(1 + \tan A)(1 + \tan B) = 2$, then $\tan (A + B) = 1$.

Solve the equations.

42. $\sin \theta + \sin 3\theta = 0$. 43. $\sin 7\theta - \sin \theta = \sin 3\theta$.

44. $\sin \theta + \sin 2\theta + \sin 3\theta = 0$. 45. $\sin \theta + \sin 9\theta = \sin 3\theta + \sin 7\theta$.

46. $\cos 2\theta - \cos 4\theta = \sin\theta$. 47. $\sin^2\theta + \sin^2 2\theta = 1$.

48. $\sin\alpha + \sin(\theta - \alpha) + \sin(2\theta + \alpha) = \sin(\theta + \alpha) + \sin(2\theta - \alpha)$.

49. If $\dfrac{\sin x}{a_1} = \dfrac{\sin 3x}{a_3} = \dfrac{\sin 5x}{a_5}$, then $\dfrac{a_1 - 2a_3 + a_5}{a_3} = \dfrac{a_3 - 3a_1}{a_1}$.

50. If $\dfrac{\cos x}{a_1} = \dfrac{\cos(x+\theta)}{a_2} = \dfrac{\cos(x+2\theta)}{a_3} = \dfrac{\cos(x+3\theta)}{a_4}$,

prove
$$\frac{a_1 + a_3}{a_2} = \frac{a_2 + a_4}{a_3}.$$

Prove the following identities:

51. $\sin(36° + A) - \sin(36° - A) - \sin(72° + A) + \sin(72° - A)$
$= \sin A$.

52. $\sin(54° + A) + \sin(54° - A) - \sin(18° + A) - \sin(18° - A)$
$= \cos A$.

53. $\sin\alpha\sin(\beta - \gamma) + \sin\beta\sin(\gamma - \alpha) + \sin\gamma\sin(\alpha - \beta) = 0$.

54. $\cos\alpha\sin(\beta - \gamma) + \cos\beta\sin(\gamma - \alpha) + \cos\gamma\sin(\alpha - \beta) = 0$.

55. $\sin(\alpha - \beta) + \sin(\beta - \gamma) + \sin(\gamma - \alpha)$
$= -4\sin\dfrac{\alpha - \beta}{2}\sin\dfrac{\beta - \gamma}{2}\sin\dfrac{\gamma - \alpha}{2}$.

56. $\sin(\alpha + \beta - \gamma) + \sin(\beta + \gamma - \alpha) + \sin(\gamma + \alpha - \beta) - \sin(\alpha + \beta + \gamma)$
$= 4\sin\alpha\sin\beta\sin\gamma$.

57. $\cos(\alpha + \beta - \gamma) + \cos(\beta + \gamma - \alpha) + \cos(\gamma + \alpha - \beta) + \cos(\alpha + \beta - \gamma)$
$= 4\cos\alpha\cos\beta\cos\gamma$.

58. $\sin\alpha\operatorname{cosec}(\alpha - \beta)\operatorname{cosec}(\alpha - \gamma) + \sin\beta\operatorname{cosec}(\beta - \gamma)\operatorname{cosec}(\beta - \alpha)$
$+ \sin\gamma\operatorname{cosec}(\gamma - \alpha)\operatorname{cosec}(\gamma - \beta) = 0$.

59. $\cos^2 x + \cos^2 y + \cos^2 z + \cos^2(x + y + z)$
$= 2\{1 + \cos(y + z)\cos(z + x)\cos(x + y)\}$.

60. $\sin^2 x + \sin^2 y + \sin^2 z + \sin^2(x + y + z)$
$= 2\{1 - \cos(y + z)\cos(z + x)\cos(x + y)\}$.

CHAPTER IX.

MULTIPLE AND SUBMULTIPLE ANGLES.

125. When the ratios of an angle are known, we have already shown how to derive the ratios of half, or double, that angle; providing the larger angle is less than two right angles. We shall now show those results to be universally true, and give others of a similar character. For this purpose we shall employ the values of $\sin(A+B)$ and $\cos(A+B)$ from Art. 114. These formulæ being universally true, any results algebraically deduced from them must also be universally true.

126. *To express the ratios of* 2A *in terms of the ratios of* A.

From Art. 114 we have

$$\sin(A+B) = \sin A \cos B + \cos A \sin B. \quad (1)$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B. \quad (2)$$

In each of these formulæ, for B write A.

then $$\sin(A+A) = \sin A \cos A + \cos A \sin A,$$

or $$\sin 2A = 2 \sin A \cos A. \quad (3)$$

Also $$\cos(A+A) = \cos A \cos A - \sin A \sin A,$$

or $$\cos 2A = \cos^2 A - \sin^2 A \quad (4)$$

$$= 2\cos^2 A - 1 \quad (5)$$

$$= 1 - 2\sin^2 A. \quad (6)$$

Again $$\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{2 \sin A \cos A}{\cos^2 A - \sin^2 A} = \frac{2 \tan A}{1 - \tan^2 A}. \quad (7)$$

The last step being obtained by dividing numerator and denominator of the previous fraction by $\cos^2 A$.

127. *To express the ratios of* 3A *in terms of the ratios of* A.

$$\begin{aligned}\sin 3A &= \sin(2A + A)\\ &= \sin 2A \cos A + \cos 2A \sin A\\ &= (2\sin A\cos A)\cos A + (1 - 2\sin^2 A)\sin A\\ &= 2\sin A\cos^2 A + \sin A - 2\sin^3 A\\ &= 2\sin A(1-\sin^2 A) + \sin A - 2\sin^3 A\\ &= 3\sin A - 4\sin^3 A. \qquad (1)\end{aligned}$$

$$\begin{aligned}\cos 3A &= \cos(2A + A)\\ &= \cos 2A\cos A - \sin 2A\sin A\\ &= (2\cos^2 A - 1)\cos A - (2\sin A\cos A)\sin A\\ &= 2\cos^3 A - \cos A - 2\sin^2 A\cos A\\ &= 2\cos^3 A - \cos A - 2(1-\cos^2 A)\cos A\\ &= 4\cos^3 A - 3\cos A. \qquad (2)\end{aligned}$$

$$\tan 3A = \frac{\sin 3A}{\cos 3A} = \frac{3\sin A - 4\sin^3 A}{4\cos^3 A - 3\cos A}$$

$$= \frac{\dfrac{3\tan A}{\cos^2 A} - 4\tan^3 A}{4 - \dfrac{3}{\cos^2 A}} = \frac{3\tan A\sec^2 A - 4\tan^3 A}{4 - 3\sec^2 A}$$

$$= \frac{3\tan A(1+\tan^2 A) - 4\tan^3 A}{4 - 3(1+\tan^2 A)} = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}. \qquad (3)$$

128. The process of Arts. 126, 127 may evidently be continued so as to express the sine or the cosine of any multiple angle in terms of the powers of sines and cosines of the single angle. It will be observed in each case that the degree of the resulting expression is always equal to the coefficient of the multiple angle. Thus the ratios of $2A$ are replaced by expressions of two dimensions, those of $3A$ by expressions of three dimensions in $\sin A$, or $\cos A$. Conversely, the square or cube of a sine or cosine may be replaced by sines or cosines of multiple angles.

129. The ratios of multiples of π and $\frac{\pi}{2}$ are sometimes useful. The truth of the following will be readily perceived, in which n denotes any integer.

1. $\sin n\pi = 0$.
2. $\cos n\pi = (-1)^n$.
3. $\sin (2n+1)\frac{\pi}{2} = (-1)^n$.
4. $\cos (2n+1)\frac{\pi}{2} = 0$.

From these the values of the other ratios may be easily obtained.

130. *To find* $\sin (n\pi + \alpha)$ *and* $\cos (n\pi + \alpha)$ *where* n *is any integer.*

We have $\sin (n\pi + \alpha) = \sin n\pi \cos \alpha + \cos n\pi \sin \alpha$ Art. 114.

$= (-1)^n \sin \alpha$. Art. 129.

Similarly $\cos (n\pi + \alpha) = \cos n\pi \cos \alpha - \sin n\pi \sin \alpha$ Art. 114.

$= (-1)^n \cos \alpha$. Art. 129.

131. *Given* $\sin A = \sin B$, *to compare* $\sin mA$ *and* $\sin mB$, *where* m *is any integer.*

Let a denote the smallest positive angle whose sine equals that of A or B.

Then $A = n\pi + (-1)^n a,\ B = r\pi + (-1)^r a$,

now $\sin (-1)^n a = (-1)^n \sin a$. Art. 101.

Therefore $\sin mA = \sin \{mn\pi + (-1)^n ma\}$

$= (-1)^{mn+n} \sin ma$. Art. 130.

Similarly $\sin mB = (-1)^{mr+r} \sin ma$.

The exponents of (-1) in the two cases are $(m+1)n$ and $(m+1)r$. If $m+1$ be even, then both exponents are even, and $\sin mA = \sin mB$, but if $m+1$ be odd, the exponents are odd or even, according as n and r are odd or even, and

$$\sin mA = (-1)^{n+r} \sin mB.$$

132. *To express the square or the cube of a sine or a cosine in terms of the sine or the cosine of a multiple angle.*

From the equations

$$\cos 2A = 2\cos^2 A - 1 \qquad (1)$$
$$= 1 - 2\sin^2 A \qquad (2)$$

we get by rearranging

$$\cos^2 A = \frac{1 + \cos 2A}{2} \qquad (3)$$

and

$$\sin^2 A = \frac{1 - \cos 2A}{2}. \qquad (4)$$

Similarly from the equations,

$$\sin 3A = 3\sin A - 4\sin^3 A, \qquad (5)$$
$$\cos 3A = 4\cos^3 A - 3\cos A, \qquad (6)$$

we get

$$\sin^3 A = \tfrac{1}{4}(3\sin A - \sin 3A), \qquad (7)$$

and

$$\cos^3 A = \tfrac{1}{4}(3\cos A + \cos 3A). \qquad (8)$$

By means of these formulæ any power of a sine or a cosine, whose exponent contains no factor except 2 or 3, may be replaced by sines or cosines of multiple angles.

133. In transforming trigonometrical expressions it is usually easier to work with multiple angles than powers of the sine or cosine. We give a few simple examples.

Ex. 1.—$\text{Sin}^2 A + \sin^2(A + 60°) + \sin^2(A + 120°)$

$= \frac{1}{2}\{1 - \cos 2A + 1 - \cos(2A + 120°) + 1 - \cos(2A + 240°)\}$ Art. 132.

$= \frac{1}{2}[3 - \{\cos 2A + \cos(2A + 240°)\} - \cos(2A + 120°)]$

$= \frac{1}{2}\{3 - 2\cos(2A + 120)\cos 120° - \cos(2A + 120°)\}$

$= \frac{3}{2}$, since $\cos 120° = -\frac{1}{2}$.

Ex. 2.—$\text{Sin}\, 3A \sin^3 A + \cos 3A \cos^3 A$

$= \frac{1}{4}\{\sin 3A(3\sin A - \sin 3A) + \cos 3A(3\cos A + \cos 3A)\}$ Art. 132.

$= \frac{1}{4}\{3(\sin 3A \sin A + \cos 3A \cos A) + (\cos^2 3A - \sin^2 3A)\}$

$= \frac{1}{4}\{3\cos 2A + \cos 6A\}$

$= \cos^3 2A.$ Art. 132.

EXERCISE XIX.

Prove the following identities : $2\cos^2 2A-1$

1. $\sin 4A = 2\sin 2A \cos 2A$.
2. $\cos 4A = 1 - 2\cos^2 2A$.
3. $\tan A = \dfrac{\sin 2A}{1+\cos 2A}$.
4. $\cot A = \dfrac{\sin 2A}{1-\cos 2A}$.
5. $\cot 2A = \dfrac{\cot^2 A - 1}{2\cot A}$.
6. $\tan^2 A = \dfrac{1-\cos 2A}{1+\cos 2A}$.
7. $\sec 2A = \dfrac{\sec^2 A}{1-\tan^2 A}$.
8. $\tan 2A = \dfrac{2}{\cot A - \tan A}$.
9. $\tan 3A = \dfrac{3-\tan^2 A}{\cot A - 3\tan A}$.
10. $\cot 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$.
11. $\cos 3A = (2\cos 2A - 1)\cos A$.
12. $\sin 3A = (2\cos 2A + 1)\sin A$.
13. $\dfrac{\sin A + \sin 3A}{\cos A + \cos 3A} = \tan 2A$.
14. $\dfrac{1-2\sin^2 A}{1+\sin 2A} = \dfrac{1}{\sec 2A + \tan 2A}$.
15. $\dfrac{\cos A + \sin A}{\cos A - \sin A} = \tan 2A + \sec 2A$.
16. $\tan^2 A + \cot^2 A = 2(1 + 2\cot^2 2A)$.
17. $\sin 3A \operatorname{cosec} A - \cos 3A \sec A = 2$.
18. $3\sin A - \sin 3A = 2\sin A\,(1-\cos 2A)$.
19. $3\cos A + \cos 3A = 2\cos A\,(1+\cos 2A)$.
20. $\cos 3A + \sin 3A = (\cos A - \sin A)(1 + 2\sin 2A)$.
21. $\cos 3A - \sin 3A = (\cos A + \sin A)(1 - 2\sin 2A)$.
22. $\cos A\,(1 - \tan 2A \tan A) = \cos 3A\,(1 + \tan 2A \tan A)$.
23. $\dfrac{\sin^2 2A - 4\sin^2 A}{\sin^2 2A + 4\sin^2 A - 4} = \dfrac{\cos^2 2A - 4\cos^2 A + 3}{\cos^2 2A + 4\cos^2 A - 1} = \tan^4 A$.
24. $4\sin A \sin(60^\circ - A)\sin(60^\circ + A) = \sin 3A$.
25. $4\cos A \cos(60^\circ - A)\cos(60^\circ + A) = \cos 3A$.
26. $\tan A \tan(60^\circ - A)\tan(60^\circ + A) = \tan 3A$.
27. $\tan A + \tan(60^\circ + A) + \tan(120^\circ + A) = 3\tan 3A$.
28. $\cot A + \cot(60^\circ + A) + \cot(120^\circ + A) = 3\cot 3A$.

29. $\dfrac{\cos(x-3y)-\cos(3x-y)}{\sin 2x+\sin 2y}=2\sin(x-y)$.

30. $\dfrac{\sin(x+3y)+\sin(3x+y)}{\sin 2x+\sin 2y}=2\cos(x+y)$.

31. $\dfrac{\tan\theta+\cot\theta+2}{\tan\theta+\cot\theta-2}=\dfrac{\sin^2(\theta+45^\circ)}{\sin^2(\theta-45^\circ)}$.

32. $1-\cos 3x=(1-\cos x)(1+2\cos x)^2$.

33. $\cos 9x+3\cos 7x+3\cos 5x+\cos 3x=8\cos^3 x\cos 6x$

34. $\cos 6\theta=32\cos^6\theta-48\cos^4\theta+18\cos^2\theta-1$.

35. $\cos 6\theta=1-18\sin^2\theta+48\sin^4\theta-32\sin^6\theta$

36. $\sin 5\theta=5\sin\theta-20\sin^3\theta+16\sin^5\theta$.

37. $\sin^4\theta=\frac{1}{8}\{3-4\cos 2\theta+\cos 4\theta\}$.

38. $\cos^4\theta=\frac{1}{8}\{3+4\cos 2\theta+\cos 4\theta\}$.

39. $\cos 5\theta=16\cos^5\theta-20\cos^3\theta+5\cos\theta$.

40. $\sin 6\theta=2\cos\theta\{3-16\sin^2\theta-16\sin^4\theta\}$.

41. $(\cos A+\sin A)^4+(\cos A-\sin A)^4=3-\cos 4A$.

42. $\sin 3A\sin^3 A+\cos 3A\cos^3 A=\cos^3 2A$.

43. $\sin 3A\cos^3 A+\cos 3A\sin^3 A=\frac{3}{4}\sin 4A$.

44. Given $(1+e\cos\theta)(1+e\cos\phi)=1-e^2$, prove

$$\cos\theta=\frac{-e-\cos\phi}{1+e\cos\phi},\ \text{and}\ \tan\frac{\theta}{2}\tan\frac{\phi}{2}=\sqrt{\frac{1+e}{1-e}}.$$

45. If $\sin B$ be the geometric mean of $\sin A$ and $\cos A$, then $\cos 2B=2\cos^2(A+45^\circ)$.

46. If $\sin(\alpha+\beta)\cos\gamma=\sin(\alpha+\gamma)\cos\beta$, then either $\alpha=(2n+1)\dfrac{\pi}{2}$, or $\beta-\gamma=n\pi$.

47. If $a\sin\theta+b\cos\theta=c=a\operatorname{cosec}\theta+b\sec\theta$, show that

$$\sin 2\theta=\frac{2ab}{c^2-a^2-b^2}.$$

48. If A, B, C are the angles of a triangle and m any integer, then $\sin mA=(-1)^{m+1}\sin m(B+C)$, and $\cos mA$ $=(-1)^m\cos m(B+C)$.

134. *To express* $\cos\frac{A}{2}$ *and* $\sin\frac{A}{2}$ *in terms of* $\cos A$ *for all values of* A.

In Art. 126 change A into $\frac{A}{2}$, and we obtain

$$\cos A = 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2}, \qquad (1)$$

from which

$$\cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}}, \ \sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}}. \qquad (2)$$

The exact meaning of these and similar equations in which radicals appear must be carefully observed. Any given angle has but one value for its sine or its cosine. But since every numerical quantity has two square roots, the above equations assert that for any *one* value of $\cos A$ there are *two* values each for $\cos\frac{A}{2}$ and $\sin\frac{A}{2}$. The explanation is found in the fact that the successive angles which have all the same cosine, when divided by 2, produce a series which have not the same cosine. If the value of $\cos A$ alone be given we shall be unable to determine which sign to take with the radical quantity. But if the value of A be known, or if we merely know the quadrant in which $\frac{A}{2}$ lies, we can immediately prefix the proper sign.

135. A numerical illustration will render the preceding article more easily intelligible. If we make $\cos A = \frac{1}{2}$ we get

$$\cos\frac{A}{2} = \sqrt{\frac{1}{2}\left(1+\frac{1}{2}\right)} = \frac{\sqrt{3}}{2}, \ \sin\frac{A}{2} = \sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} = \frac{1}{\sqrt{2}}.$$

Now, in making $\cos A = \frac{1}{2}$ we simply assert that

$A =$ some one of the angles 60°, 300°, 420°, 660°, etc.

$\therefore \quad \frac{A}{2} =$ some one of the angles 30°, 150°, 210°, 330°, etc.

and the above equations assert that the cosine of any angle in the latter series is either $\frac{+\sqrt{3}}{2}$ or $\frac{-\sqrt{3}}{2}$, and that its sine is either $+\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$. In any particular case we can easily determine which sign to take. For example, $\sin 210° = -\frac{1}{\sqrt{2}}$ since the sines of all angles in the third quadrant are negative.

136. *To express* $\cos\frac{A}{2}$ *and* $\sin\frac{A}{2}$ *in terms of* $\sin A$.

We have $\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)^2 = \cos^2\frac{A}{2}+\sin^2\frac{A}{2}+2\sin\frac{A}{2}\cos\frac{A}{2}$

$$= 1+\sin A.$$

Therefore $\qquad \cos\frac{A}{2}+\sin\frac{A}{2}=\sqrt{1+\sin A}. \qquad (1)$

Similarly $\qquad \cos\frac{A}{2}-\sin\frac{A}{2}=\sqrt{1-\sin A}. \qquad (2)$

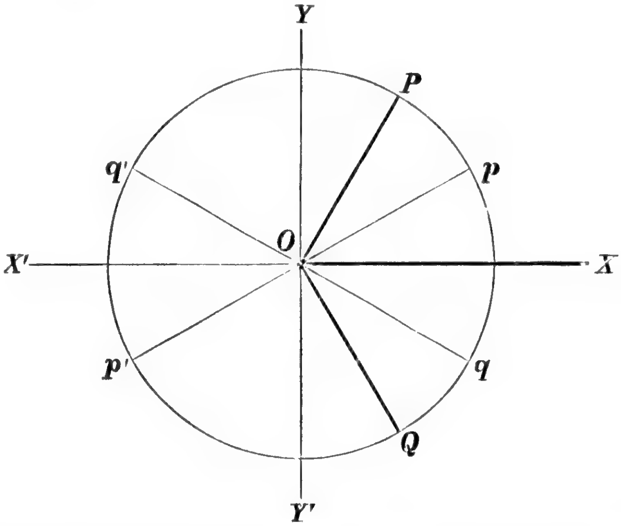
From which by addition and subtraction we obtain

$$2\cos\frac{A}{2}=\sqrt{1+\sin A}+\sqrt{1-\sin A} \qquad (3)$$

$$2\sin\frac{A}{2}=\sqrt{1+\sin A}-\sqrt{1-\sin A}. \qquad (4)$$

Now since the numerical values of the surd expressions in (1) and (2) may be taken as either positive or negative, we shall have, in (3) and (4), four different values each for $\cos\frac{A}{2}$ and $\sin\frac{A}{2}$ for each value of $\sin A$. The signs which connect the surds in (3) and (4) denote the operation of addition or subtraction, and give no indication whether the following root is to be positive or negative. This must be determined by the magnitude of the given angle as shown in Art. 139.

137. *To prove geometrically that for each value of* cos A *there are two values each for* cos $\frac{A}{2}$ *and* sin $\frac{A}{2}$.



Let XOP, XOQ be the smallest angles, positive and negative, which have the given cosine. Bisect these angles by Op, Oq, and produce these bisecting lines backwards as in the figure. Denote the angle XOP by a, and the series of positive angles having the same cosine by A.

Then $\quad A = a, \quad 2\pi - a, \quad 2\pi + a, \quad 4\pi - a$, etc.

$$\therefore \quad \frac{A}{2} = \frac{a}{2}, \quad \pi - \frac{a}{2}, \quad \pi + \frac{a}{2}, \quad 2\pi - \frac{a}{2}, \text{ etc.}$$

$$= XOp, \quad XOq', \quad XOp', \quad XOq, \text{ etc.}$$

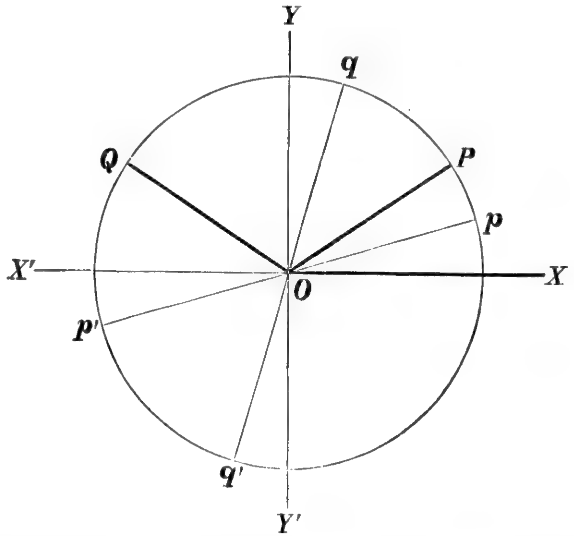
Now $\quad \cos XOp = -\cos XOq' = -\cos XOp' = \cos XOq$,

and $\quad \sin XOp = \sin XOq' = -\sin XOp' = -\sin XOq$.

Hence $\cos\frac{A}{2} = \pm\cos\frac{a}{2}$, $\sin\frac{A}{2} = \pm\sin\frac{a}{2}$, which give the two values required.

The symmetry of the figure shows the same results to be true for negative angles. The proposition is, therefore, universally true.

138. *To prove geometrically that for each value of sin A there are four values each for cos $\frac{A}{2}$ and sin $\frac{A}{2}$.*



Let XOP, XOQ be the angles in the first revolution which have the given sine. Bisect these angles by Op and Oq, and produce the bisectors backwards as in the figure. Denote XOP by α, and the series of positive angles having the same sine as XOP by A.

Then $\quad A = \alpha, \quad \pi - \alpha, \quad 2\pi + \alpha, \quad 3\pi - \alpha, \quad 4\pi + \alpha$, etc.

$$\therefore \quad \frac{A}{2} = \frac{\alpha}{2}, \; \frac{\pi}{2} - \frac{\alpha}{2}, \; \pi + \frac{\alpha}{2}, \; 3\frac{\pi}{2} - \frac{\alpha}{2}, \; 2\pi + \frac{\alpha}{2}, \text{ etc.}$$

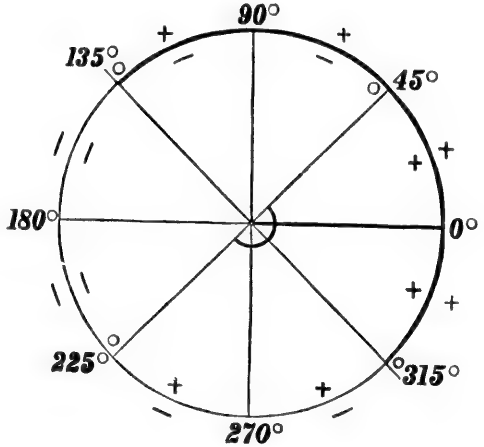
$$= XOp, \; XOq, \; XOp', \; XOq', \; XOp, \text{ etc.}$$

Now it is evident from the figure that the angles XOp, qOY, $X'Op'$, $q'OY'$, are geometrically equal; from which it easily follows that

$$\cos XOp = \sin XOq = -\cos XOp' = -\sin XOq',$$
$$\sin XOp = \cos XOq = -\sin XOp' = -\cos XOq'.$$

Hence $\cos\frac{A}{2} = \pm\cos\frac{\alpha}{2}$, or $\pm\sin\frac{\alpha}{2}$; $\sin\frac{A}{2} = \pm\sin\frac{\alpha}{2}$, or $\pm\cos\frac{\alpha}{2}$, which give the four values required.

139. *To determine the signs which must be taken with the radicals in obtaining the values of* $\cos\frac{A}{2}$ *and* $\sin\frac{A}{2}$ *from the value of* $\sin A$.



Draw a circle, divide it into eight equal sectors as in the figure. As the angle $\frac{A}{2}$ is placed in each division in succession, consider the values of $\cos\frac{A}{2}$ and $\sin\frac{A}{2}$ separately, whether positive or negative, and which is numerically the greater, and thus determine the sign of the combined expressions $\cos\frac{A}{2}+\sin\frac{A}{2}$ and $\cos\frac{A}{2}-\sin\frac{A}{2}$. If we mark the sign of the former on the outside of the circle, and that of the latter on the inside, the result will be as shown in the figure. The signs thus obtained for any sections of the figure are those which must be taken with the corresponding radicals when $\frac{A}{2}$ lies in that section.

The reader should carefully note the points for which the two expressions are zero, and where they consequently change sign.

140. The results of Art. 139 may also be obtained symbolically, and it is instructive to compare the two processes.

Let a be the smallest positive angle whose cosine equals $\cos A$.

Then
$$A = 2n\pi \pm a,$$

and
$$\cos\frac{A}{2} = \cos\left(n\pi \pm \frac{a}{2}\right)$$
$$= \cos n\pi \cos\frac{a}{2} \mp \sin n\pi \sin\frac{a}{2}$$
$$= (-1)^n \cos\frac{a}{2}. \qquad (1)$$

Also
$$\sin\frac{A}{2} = \sin\left(n\pi \pm \frac{a}{2}\right)$$
$$= \sin n\pi \cos\frac{a}{2} \pm \cos n\pi \sin\frac{a}{2}$$
$$= \pm(-1)^n \sin\frac{a}{2}. \qquad (2)$$

Equations (1) and (2) show that for each value of $\cos A$ there are two values each for $\cos\frac{A}{2}$ and $\sin\frac{A}{2}$; they also determine the sign to be prefixed.

Again, let a denote the smallest positive angle whose sine equals $\sin A$.

Then
$$A = n\pi + (-1)^n a.$$

We must now distinguish the cases in which n is even or odd.

1. Let n be even and equal $2m$.

Then
$$A = 2m\pi + a,$$

and
$$\cos\frac{A}{2} = \cos\left(m\pi + \frac{a}{2}\right)$$
$$= (-1)^m \cos\frac{a}{2}. \qquad (3)$$

Similarly
$$\sin\frac{A}{2} = \sin\left(m\pi + \frac{a}{2}\right)$$
$$= (-1)^m \cos\frac{a}{2}. \qquad (4)$$

2. Let n be odd and equal $2m+1$.

Then
$$A = (2m+1)\pi - \alpha,$$

and
$$\cos\frac{A}{2} = \cos\left(m\pi + \frac{\pi-\alpha}{2}\right)$$
$$= (-1)^m \cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)$$
$$= (-1)^m \sin\frac{\alpha}{2}. \tag{5}$$

Similarly
$$\sin\frac{A}{2} = \sin\left(m\pi + \frac{\pi-\alpha}{2}\right)$$
$$= (-1)^m \sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)$$
$$= (-1)^m \cos\frac{\alpha}{2}. \tag{6}$$

Equations (3)....(6) show that for each value of $\sin A$ there are four values each for $\cos\frac{A}{2}$ and $\sin\frac{A}{2}$, and they also determine the sign to be prefixed in each case.

141. In the previous article α denotes an angle in the first quadrant, $\frac{\alpha}{2}$ lies between $0°$ and $45°$, $\cos\frac{\alpha}{2}$ and $\sin\frac{\alpha}{2}$ are both positive, $\cos\frac{\alpha}{2} > \sin\frac{\alpha}{2}$, hence equations (3) and (4), Art. 136, give

$$2\cos\frac{\alpha}{2} = \sqrt{1+\sin A} + \sqrt{1-\sin A}, \tag{1}$$

$$2\sin\frac{\alpha}{2} = \sqrt{1+\sin A} - \sqrt{1-\sin A}. \tag{2}$$

Then equations (3) and (4), Art. 140, give

$$\cos\frac{A}{2} = (-1)^m\cos\frac{\alpha}{2} = \frac{(-1)^m}{2}\left\{\sqrt{1+\sin A} + \sqrt{1-\sin A}\right\} \tag{3}$$

$$\sin\frac{A}{2} = (-1)^m\sin\frac{\alpha}{2} = \frac{(-1)^m}{2}\left\{\sqrt{1+\sin A} - \sqrt{1-\sin A}\right\} \tag{4}$$

in which n is even, whilst (5) and (6) show that the connecting signs on the right must be reversed when n is odd. In both cases m is the integral part of $\frac{n}{2}$.

Ex.—Let $A = 500° = 3\pi - 40°$; then n is odd and $m = 1$.

Then $\cos\frac{A}{2} = (-1)^m \sin\frac{a}{2} = -\frac{1}{2}\left\{\sqrt{1+\sin A} - \sqrt{1-\sin A}\right\}$.

This result should be verified by reference to the diagram of Art. 139.

142. *To express* $\tan\frac{A}{2}$ *in terms of* $\tan A$.

In Art. 126, change A into $\frac{A}{2}$ and we have,

$$\tan A = \frac{2\tan\frac{1}{2}A}{1-\tan^2\frac{1}{2}A}.$$

Then $$\tan A \cdot \tan^2\tfrac{1}{2}A + 2\tan\tfrac{1}{2}A - \tan A = 0,$$

from which $$\tan\frac{A}{2} = \frac{-1 \pm \sqrt{1+\tan^2 A}}{\tan A}.$$

Thus, for each value of $\tan A$ there are two different numerical values for $\tan\frac{A}{2}$. The student should draw a diagram illustrating the double value, and also examine the result symbolically, as has been previously done in the case of the sine and cosine.

143. The formulæ of this chapter may be employed to obtain the ratios of the angles formerly obtained by geometrical constructions.

Ex. 1.—To find the ratios of 45°.

Since 45° is its own complement, we have

$$\sin^2 45° = \cos^2 45° = 1 - \sin^2 45°.$$

Therefore $$2\sin^2 45° = 1, \text{ or } \sin 45° = \cos 45° = \frac{1}{\sqrt{2}}.$$

Ex. 2.—To find the ratios of 30° and 60°.

Since 30° and 60° are complementary angles, we have

$$\cos 30^\circ = \sin 60^\circ = 2 \sin 30^\circ \cos 30^\circ.$$

Therefore $$\cos 30^\circ (1 - 2 \sin 30^\circ) = 0.$$

Now $\cos 30^\circ$ is not zero, $\therefore \sin 30^\circ = \frac{1}{2}$.

Ex. 3.—To find the ratios of 18°.

Since 36° and 54° are complementary angles, we have

$$\sin 36^\circ = \cos 54^\circ.$$

Therefore $2 \sin 18^\circ \cos 18^\circ = 4 \cos^3 18^\circ - 3 \cos 18^\circ$. Art. 127.

Divide by $\cos 18^\circ$, $$2 \sin 18^\circ = 4 \cos^2 18^\circ - 3$$
$$= 1 - 4 \sin^2 18^\circ.$$

Therefore $$4 \sin^2 18^\circ + 2 \sin 18^\circ - 1 = 0.$$

Solving for $\sin 18^\circ$, $$\sin 18^\circ = \frac{\sqrt{5} - 1}{4}.$$

144. In Art. 127, change A into $\frac{A}{3}$ and we get

$$\sin A = 3 \sin \frac{A}{3} - 4 \sin^3 \frac{A}{3}.$$

Let $\sin A = a$, for $\sin \frac{A}{3}$ write x, and rearrange the result; we get $$4x^3 - 3x^2 + a = 0.$$

Hence it appears that $\sin \frac{A}{3}$ may be found from $\sin A$ by the solution of a cubic equation. Similarly, $\cos \frac{A}{3}$ may be found from the value of $\cos A$. If to a we give any particular numerical value the roots of the resulting equation can always be found. The process, however, does not belong to Elementary Algebra, and is consequently seldom employed. Other methods are adopted for obtaining the numerical values of the ratios, and when these have been obtained they may be used to solve cubic equations.

It is instructive to observe that the bisection of an angle and the solution of a quadratic equation are corresponding operations, and both are possible by elementary means. Also, the trisection of an angle and the solution of a cubic equation correspond, but neither is possible by elementary methods.

EXERCISE XX.

1. Apply the formulæ of Art. 134 to find $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ for the following values of A:

(1) 30°. (2) −30°. (3) 315°. (4) 378°. (5) 432°.

2. Verify the results of the preceding example by the method of Art. 136.

3. Find all the ratios of 9° from the value of sin 18°.

4. Given $\tan 15° = 2 - \sqrt{3}$, find $\tan 7\frac{1}{2}°$ and $\tan 37\frac{1}{2}°$.

5. Given $\tan 2A = -\frac{24}{7}$, find $\sin A$ and $\cos A$. Draw a figure showing how the different values are possible.

6. Given $2\sin A = \sqrt{1+\sin 2A} + \sqrt{1-\sin 2A}$, find the limits between which A lies.

7. Given $2\cos A = -\sqrt{1+\sin 2A} + \sqrt{1-\sin 2A}$, find the limits between which A lies.

8. Prove $\sec A = \dfrac{\cos\frac{1}{2}A}{\sqrt{1+\sin A}} + \dfrac{\sin\frac{1}{2}A}{\sqrt{1-\sin A}}$, and show how to determine the correct signs for the radicals.

9. Prove $\cos\dfrac{A}{2^n} = \dfrac{1}{2}\sqrt{2+\sqrt{2+\ldots\sqrt{2+2\cos A}}}$ which contains n radicals.

10. Prove $\sin\dfrac{A}{2^n} = \dfrac{1}{2}\sqrt{2-\sqrt{2+\ldots\sqrt{2+2\cos A}}}$ which contains n radicals, and all connecting signs except the first being positive.

11. Adapt the two preceding examples to find the perimeter and area of a regular polygon of $2n$ sides, and thence show how the value of π may be found.

12. Prove $(\cos A + \cos B)^2 + (\sin A + \sin B)^2 = 4 \cos^2 \frac{A - B}{2}$.

13. Prove $(\cos A + \cos B)^2 - (\sin A + \sin B)^2 = 4 \cos (A + B) \cos^2 \frac{A - B}{2}$.

14. Obtain other formulæ similar to the two preceding by changing one or more of the connecting signs.

15. Express $\cot \frac{\theta}{2}$ in terms of $\cot \theta$, and thence show that $\cot \frac{\theta}{2} > 1 + \cot \theta$ for all values of θ from 0 to π.

16. Given $\tan x = (2 + \sqrt{3}) \tan \frac{x}{3}$, find $\tan x$.

17. Prove $\tan 142\frac{1}{2}^\circ = 2 + \sqrt{2} - \sqrt{3} - \sqrt{6}$ in three different ways.

18. Adapt the method of Art. 143 to find $\cos 18^\circ$.

19. Assume $A = 180^\circ - 2A$, and thence determine the ratios for 60°

20. Assume $5A = 180^\circ$, and thence find the ratios for 36°.

21. In Art. 143, Ex. 3, if the negative sign be taken with the radical, of what angle will the resulting expression be the sine?

22. If $\cos a$ be one root of the equation $4x^3 - 3x = a$, then $\cos (120^\circ + a)$ and $\cos (120^\circ - a)$ are the other roots.

23. From the preceding example deduce

(1) $\cos a + \cos (120^\circ + a) + \cos (120^\circ - a) = 0$.

(2) $\cos a \cos (120^\circ + a) + \cos (120^\circ + a) \cos (120^\circ - a) + \cos (120^\circ - a) \cos a = -\frac{3}{4}$.

(3) $\cos a \cos (120^\circ + a) \cos (120^\circ - a) = \frac{1}{4} \cos 3a$.

24. Prove the three preceding identities trigonometrically, and thence write the equation whose roots are $\cos a$, $\cos (120^\circ + a)$, $\cos (120^\circ - a)$.

CHAPTER X.

INVERSE NOTATION AND SUBSIDIARY ANGLES.

145. In the equation $\tan\theta = a$, there are two quantities involved, θ representing an angle, and a which represents a number. The relation existing between these quantities is indicated by the symbol "tan" prefixed to the angle, whilst the number stands alone. We might, with equal propriety, place the symbol of relation before the number, and let the angle stand alone. In practical work the latter method is frequently the more convenient. To avoid confusion the ratio symbol is marked with a negative sign when placed before the number. Thus,

$$\tan\theta = a \text{ and } \theta = \tan^{-1} a.$$

Both express the same relation between θ and a, *viz.*, θ is an angle of which a is the tangent. The latter expression has the decided advantage of representing both the angle and the numerical value of its tangent by a single symbol. Similar remarks apply to all the trigonometrical ratios.

146. The symbols sin, cos, tan, etc., denote the operation of passing from an angle to certain numbers dependent on it; whilst $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, etc., denote the inverse operation of returning from one of those numbers to the original angle. For this reason they are called **inverse symbols.** For a similar reason they are sometimes called **anti-functions.**

147. The numerical measure of an angle may be represented by any symbol denoting a number; the trigonometrical ratios denote numbers, and consequently such expressions as $\sin(\sin x)$, $\tan(\cot x)$, etc., occasionally present themselves in mathematics. Their treatment requires nothing further than careful attention to their meaning.

148. The reader should endeavor to acquire facility in the use of the new notation by translating into it a considerable number of the formulæ already obtained. We give a few simple examples.

Ex. 1.—Express the relation $\tan\theta = \frac{1}{\cot\theta}$ in the inverse notation.

Let $$\tan\theta = \frac{a}{b}, \text{ or } \theta = \tan^{-1}\frac{a}{b},$$

then $$\cot\theta = \frac{b}{a}, \text{ or } \theta = \cot^{-1}\frac{b}{a}.$$

Therefore, $\tan^{-1}\frac{a}{b} = \cot^{-1}\frac{b}{a}$ is the expression required.

Ex. 2.—Express the relation $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$ in the inverse notation.

Let $$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right) = a,$$

then $$\theta = \tan^{-1}a, \text{ and } \frac{\pi}{2} - \theta = \cot^{-1}a,$$

from which $$\tan^{-1}a + \cot^{-1}a = \theta + \frac{\pi}{2} - \theta = \frac{\pi}{2},$$

the relation required.

Ex. 3.—Given $\sin\theta = a$, express the formulæ for $\sin 2\theta$ and $\sin 3\theta$ in the inverse notation.

Since $$\sin\theta = a$$
therefore $$\cos\theta = \sqrt{1-a^2}$$
and $$\theta = \sin^{-1}a.$$

Then $$\sin 2\theta = 2\sin\theta\cos\theta,$$
therefore, $$2\theta = \sin^{-1}(2\sin\theta\cos\theta)$$
or $$2\sin^{-1}a = \sin^{-1}(2a\sqrt{1-a^2}). \qquad (1)$$

Similarly from $$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$
we get $$3\sin^{-1}a = \sin^{-1}(3a - 4a^3). \qquad (2)$$

Ex. 4.—To prove $\tan^{-1} a + \tan^{-1} b = \tan^{-1}\left(\dfrac{a+b}{1-ab}\right)$.

Let $\qquad \tan^{-1} a = A$, or $a = \tan A$;

and $\qquad \tan^{-1} b = B$, or $b = \tan B$;

then $$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B},$$

or, $$\tan(\tan^{-1} a + \tan^{-1} b) = \frac{a+b}{1-ab},$$

therefore, $$\tan^{-1} a + \tan^{-1} b = \tan^{-1}\frac{a+b}{1-ab}.$$

This formula is of very frequent application.

149. There is one point which requires careful attention.

In the equation $\tan \theta = a$, if we gave any one value to θ, we get but one value for a. For example, if $\theta = 45°$ then $a = 1$. If, however, we take the equivalent equation $\tan^{-1} a = \theta$ and give a a particular value, we get an infinite number of corresponding values for θ. Thus, if $a = 1$, then $\theta = 45°$, $225°$, or any angle denoted by $n\pi + \dfrac{\pi}{4}$ in which n represents any integer, positive or negative. In other words the ratios $\sin \theta$, $\cos \theta$, etc., are single-valued functions, whilst the anti-functions $\sin^{-1} a$, $\cos^{-1} a$, etc., are many-valued functions. A neglect to recognize this fact will at once lead to error. For example we have written $\tan^{-1} a + \cot^{-1} a = \dfrac{\pi}{2}$. Let $a = \sqrt{3}$, then for $\tan^{-1} a$ we may write $60°$, $240°$, etc., and for $\cot^{-1} a$ we may write $30°$, $210°$, etc., hence the given equation is true only for particular values of the inverse functions involved. Similar remarks apply to all the inverse functions.

150. As a further illustration of the preceding article we give the following example:

Ex.—Find the general value of $\sin^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}$.

We have
$$\sin\left(\sin^{-1}\tfrac{1}{2}+\cos^{-1}\tfrac{1}{2}\right)=\sin\left(\sin^{-1}\tfrac{1}{2}\right)\cos\left(\cos^{-1}\tfrac{1}{2}\right)+\cos\left(\sin^{-1}\tfrac{1}{2}\right)\sin\left(\cos^{-1}\tfrac{1}{2}\right)$$
$$=\tfrac{1}{2}\cdot\tfrac{1}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}$$
$$=1=\sin\frac{\pi}{2},$$

therefore
$$\sin^{-1}\tfrac{1}{2}+\cos^{-1}\tfrac{1}{2}=n\pi+(-1)^n\frac{\pi}{2}. \qquad (1)$$

Again
$$\cos\left(\sin^{-1}\tfrac{1}{2}+\cos^{-1}\tfrac{1}{2}\right)=\cos\left(\sin^{-1}\tfrac{1}{2}\right)\cos\left(\cos^{-1}\tfrac{1}{2}\right)-\sin\left(\sin^{-1}\tfrac{1}{2}\right)\sin\left(\cos^{-1}\tfrac{1}{2}\right)$$
$$=\frac{\sqrt{3}}{2}\cdot\frac{1}{2}-\frac{1}{2}\cdot\frac{\sqrt{3}}{2}$$
$$=0=\cos\frac{\pi}{2},$$

therefore
$$\sin^{-1}\tfrac{1}{2}+\cos^{-1}\tfrac{1}{2}=2n\pi\pm\frac{\pi}{2}. \qquad (2)$$

Now (1) and (2) do not give the same series of angles, hence they cannot both be correct. Upon trial it will be found that (1) gives only part of the admissible values; that (2) gives the same values together with others not admissible, though both methods give the correct result when the angles have their least positive value. Combining the general values of the angles taken singly, we have

$$\sin^{-1}\tfrac{1}{2}+\cos^{-1}\tfrac{1}{2}=n\pi+(-1)^n\frac{\pi}{6}+2m\pi\pm\frac{\pi}{3}$$
$$=(2m+n)\pi\pm\frac{\pi}{3}+(-1)^n\frac{\pi}{6}$$
$$=r\pi\pm\frac{\pi}{3}+(-1)^r\frac{\pi}{6}$$
$$=\left\{6r\pm2+(-1)^r\right\}\frac{\pi}{6}, \qquad (3)$$

which is the correct result required. It will be an instructive exercise for the student to search for the fallacy in (1) and (2). These methods give correct results when the anti-functions are restricted to their least positive values, and are frequently employed in such cases. The method (3) must be employed when the anti-functions are derived from different ratios, as in the given example.

151. In the solution of trigonometrical equations care must be taken not to omit any real solution, and to exclude all roots not belonging to the original equation. The following example will be instructive in this particular:

Ex. Solve the equation $\sin\theta+\cos\theta=\sqrt{\dfrac{3}{2}}$.

FIRST SOLUTION.

Squaring given equation we get

$$\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta=\frac{3}{2},$$

therefore $$1+\sin 2\theta=\frac{3}{2},$$

or, $$\sin 2\theta=\frac{1}{2}=\sin\frac{\pi}{6},$$

therefore $$2\theta=n\pi+(-1)^n\frac{\pi}{6}. \qquad (1)$$

SECOND SOLUTION.

Transpose $\cos\theta$ and square:

Then $$\sin^2\theta=\frac{3}{2}-\sqrt{6}\,.\cos\theta+\cos^2\theta,$$

or $$2\cos^2\theta-\sqrt{6}\,.\cos\theta+\frac{1}{2}=0,$$

therefore $$\cos\theta=\frac{\sqrt{6}\pm\sqrt{2}}{4}=\cos\frac{\pi}{12},\text{ or }\cos\frac{5\pi}{12},$$

therefore $$\theta=2n\pi\pm\frac{\pi}{12},\text{ or }2n\pi\pm\frac{5\pi}{12}. \qquad (2)$$

THIRD SOLUTION.

Transpose $\sin\theta$, square and proceed as before, and we obtain

$$\sin\theta = \frac{\sqrt{6} \pm \sqrt{2}}{4} = \sin\frac{5\pi}{12}, \text{ or } \sin\frac{\pi}{12},$$

therefore $$\theta = n\pi + (-1)^n \frac{5\pi}{12}, \text{ or } n\pi + (-1)^n \frac{\pi}{12}. \qquad (3)$$

Let us now examine the results of the three solutions and see whether the same series of angles is given in each. Taking positive angles only, we have from (1)

$$\theta = \frac{1}{2}\left\{n\pi + (-1)^n \frac{\pi}{2}\right\} = 15^\circ,\ 75^\circ,\ 195^\circ,\ 255^\circ,\ 375^\circ,\ 435^\circ.$$

From (2)

$$\theta = 2n\pi \pm \frac{\pi}{12} = 15^\circ,\ 345^\circ,\ 375^\circ \ldots.$$

or $$= 2n\pi \pm \frac{5\pi}{12} = 75^\circ,\ 285^\circ,\ 435^\circ \ldots.$$

From (3)

$$\theta = n\pi + (-1)^n \frac{5\pi}{12} = 75^\circ,\ 105^\circ,\ 435^\circ,\ 465^\circ \ldots.$$

or $$= n\pi + (-1)^n \frac{\pi}{12} = 15^\circ,\ 165^\circ,\ 375^\circ,\ 525^\circ \ldots.$$

The only angles between 0° and 360° common to the three solutions are 15° and 75°, and on testing these they are found to satisfy the original equation. Testing the other angles we find the following results :

$$\sin 195^\circ + \cos 195^\circ = -\sin 15^\circ - \cos 15^\circ = -\frac{\sqrt{3}}{2},$$

$$\sin 255^\circ + \cos 255^\circ = -\cos 15^\circ - \sin 15^\circ = -\frac{\sqrt{3}}{2}.$$

Hence these angles belong to the equation

$$\sin\theta + \cos\theta = -\frac{\sqrt{3}}{2}. \qquad (4)$$

Again $\quad \sin 285^\circ + \cos 285^\circ = -\cos 15^\circ + \sin 15^\circ$
$\quad\quad \sin 345^\circ + \cos 345^\circ = -\sin 15^\circ + \cos 15^\circ.$

Hence these angles belong respectively to the equations

$$\sin\theta - \cos\theta = \pm\sqrt{\frac{3}{2}}. \tag{5}$$

Similarly it may be shown that the angles 105°, 165° belong respectively to the equations

$$-\sin\theta + \cos\theta = \pm\sqrt{\frac{3}{2}}. \tag{6}$$

It will be found by trial that equations (4), (5) and (6), when solved by the given methods, give precisely the same results as the original equation. The roots obtained from an equation after squaring do not, in general, all belong to the original equation. (See High School Algebra, Part II., chap. XII.)

152. The preceding article shows that in solving trigonometrical functions it is desirable to avoid squaring whenever possible. We, therefore, give another solution of the same equation by a method which avoids introducing roots not belonging to the given equation.

Given equation is $\quad \sin\theta + \cos\theta = \sqrt{\frac{3}{2}}.$

Now $\quad \cos 45^\circ = \sin 45^\circ = \frac{1}{\sqrt{2}}.$

Multiplying the terms in succession by these equals we get

$$\sin\theta\cos 45^\circ + \cos\theta\sin 45^\circ = \frac{\sqrt{3}}{2},$$

or $\quad \sin(\theta + 45^\circ) = \sin 60^\circ,$

therefore $\quad \theta + 45^\circ = n\pi + (-1)^n\frac{\pi}{3}$

from which $\quad \theta = (n - \tfrac{1}{4})\pi + (-1)^n\frac{\pi}{3}.$

Upon trial it will be found that the series of angles thus obtained, but no others, satisfy the given equation.

153. An angle introduced to assist in the solution of an equation, or to resolve a given expression into factors, is called a **subsidiary angle.** A subsidiary angle was employed in solving the equation in the preceding article.

154. We now give a few examples of the use of subsidiary angles.

Ex. 1.—Solve the equation $a \cos \theta + b \sin \theta = c$.

Assume $$\frac{b}{a} = \tan \phi = \frac{\sin \phi}{\cos \phi},$$

then $$\frac{\cos \phi}{a} = \frac{\sin \phi}{b} = \frac{1}{\sqrt{a^2+b^2}}.$$

Multiplying the successive terms of the given equation by these equals, we have

$$\cos \theta \cos \phi + \sin \theta \sin \phi = \frac{c}{\sqrt{a^2+b^2}}$$

or $$\cos (\theta - \phi) = \frac{c}{\sqrt{a^2+b^2}},$$

therefore $$\theta - \phi = \cos^{-1} \frac{c}{\sqrt{a^2+b^2}}$$

and $$\theta = \cos^{-1} \frac{c}{\sqrt{a^2+b^2}} + \tan^{-1} \frac{b}{a}$$

Since the cosine of any angle is always less than unity, the equation will be impossible when $c^2 > a^2 + b^2$.

Ex. 2.—Find the value of θ for which the expression $\cos \theta + \sqrt{3} \sin \theta$ has the greatest numerical value.

We have $$\cos \theta + \sqrt{3} \sin \theta = \cos \theta + \tan \frac{\pi}{3} \sin \theta$$

$$= \frac{1}{\cos \frac{\pi}{3}} \left(\cos \theta \cos \frac{\pi}{3} + \sin \theta \sin \frac{\pi}{3} \right)$$

$$= 2 \cos \left(\theta - \frac{\pi}{3} \right).$$

Now the cosine of an angle is greatest when the angle is zero, hence $\theta = \frac{\pi}{3}$ gives the value required.

Ex. 3.—Express $a+b$ as the product of factors.

Let a be the greater of the two numbers.

Assume
$$\frac{b}{a} = \cos\phi$$

Then
$$a+b = a\left(1+\frac{b}{a}\right) = a(1+\cos\phi)$$
$$= 2a\cos^2\frac{\phi}{2} = 2a\cos^2\left(\frac{1}{2}\cos^{-1}\frac{b}{a}\right).$$

Ex. 4.—Express $a^2+b^2-2ab\cos C$ as a complete square.

$$a^2+b^2-2ab\cos C = (a+b)^2 - 2ab(1+\cos C)$$
$$= (a+b)^2\left\{1-\frac{4ab}{(a+b)^2}\cos^2\frac{C}{2}\right\}$$
$$= (a+b)^2(1-\sin^2\phi), \text{ if } \sin\phi = \frac{2\sqrt{ab}}{a+b}\cos\frac{C}{2}$$
$$= (a+b)^2\cos^2\phi.$$

In examples like the preceding, in which numerical quantities are replaced by trigonometrical ratios, we must be careful not to assume an impossibility. In Ex. 1, we are certain that there is some angle whose tangent is $\frac{b}{a}$, since the tangent of an angle may have any value. The value of ϕ will be found in any special case from the tables. In Ex. 3, it was necessary to assume $a > b$, otherwise $\cos\phi = \frac{b}{a}$ would be impossible. In Ex. 4, $(a+b)^2$ is never less than $4ab$, and $\cos^2\frac{C}{2} < 1$, hence the value given to $\sin\theta$ is always possible.

EXERCISE XXI.

1. Find the value of $\sin\left(2\sin^{-1}\frac{1}{2}\right)$, $\cos\left(\sin^{-1}\frac{1}{2}\right)$, $\tan\left(\cot^{-1}\frac{3}{4}\right)$.

2. Simplify $\sin\left(\sin^{-1}\frac{1}{\sqrt{2}}+\cos^{-1}\frac{1}{2}\right)$, $\tan\left(\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}\right)$.

3. Prove $\sin^{-1}\frac{1}{2}=\cos^{-1}\frac{\sqrt{3}}{2}=\cot^{-1}\sqrt{3}$, when each expression has its least positive value. Examine the effect of removing this restriction.

4. Find the values of $\sin\left(\frac{\pi}{2}-\cos^{-1}\frac{1}{3}\right)$ and $\tan\left(\frac{\pi}{2}+\tan^{-1}\frac{1}{5}\right)$.

5. Express the equation $\sin^{-1}x=2\cos^{-1}x$ in the ordinary notation. Also find x and $\sin^{-1}x$.

6. Prove $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$, when the angles have their least positive value. Illustrate by a diagram.

7. Show that one value of $\sin^{-1}\frac{1}{\sqrt{5}}+\cot^{-1}3$ is 45°. Find other values.

8. Prove the following when the angles have their least positive value:

(1) $2\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{5}{12}$. (2) $\tan^{-1}\frac{5}{7}+\tan^{-1}\frac{1}{6}=\frac{\pi}{4}$.

(3) $\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{3}=\frac{\pi}{4}$. (4) $\cot^{-1}\frac{3}{4}+\cot^{-1}\frac{1}{7}=\frac{3\pi}{4}$.

(5) $\cos^{-1}\frac{1}{2}+2\sin^{-1}\frac{1}{2}=\frac{2\pi}{3}$. (6) $\cos^{-1}\frac{9}{\sqrt{82}}+\operatorname{cosec}^{-1}\frac{\sqrt{41}}{4}=\frac{\pi}{4}$.

9. Prove $\tan^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}=\tan^{-1}\frac{61}{45}$, or $\tan^{-1}\frac{11}{75}$. Explain the two values by a diagram.

10. Prove $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=n\pi+\frac{\pi}{4}$.

11. Prove $\tan^{-1}a-\tan^{-1}b=\tan^{-1}\frac{a-b}{1+ab}$.

12. Prove $\tan^{-1}\dfrac{a-b}{1+ab}+\tan^{-1}\dfrac{b-c}{1+bc}+\tan^{-1}\dfrac{c-a}{1+ca}=0$, or $n\pi$.

13. Find the values of

$$\tan(\tan^{-1}x+\cot^{-1}x) \text{ and } \sin(\sin^{-1}x+\cos^{-1}x).$$

Why has the latter two values whilst the former has but one?

14. Solve the following equations:

(1) $\tan^{-1}2x+\tan^{-1}3x=\dfrac{\pi}{4}$.

(2) $5\sin^{-1}x=\cos^{-1}x$.

(3) $\sin^{-1}2x-\sin^{-1}x\sqrt{3}=\sin^{-1}x$.

(4) $\sin 2\cos^{-1}\cot 2\tan^{-1}x=0$.

(5) $\sin^{-1}x+\sin^{-1}\dfrac{x}{2}=\dfrac{\pi}{4}$.

(6) $\sin^{-1}x+\tan^{-1}x=\dfrac{\pi}{2}$.

(7) $\tan^{-1}a+\tan^{-1}b=\tan^{-1}x$.

(8) $\cot^{-1}x+\tan^{-1}x=\dfrac{\pi}{4}$.

(9) $\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x$.

15. Given $\tan A=a$, express the equality $\sin 2A=\dfrac{2\tan A}{1+\tan^2 A}$ in the inverse notation.

16. Solve $\sin^{-1}\dfrac{2a}{1+a^2}+\sin^{-1}\dfrac{2b}{1+b^2}=2\tan^{-1}x$.

17. By definition $\tan(\tan^{-1}a)=a$, and $\tan^{-1}(\tan a)=a$. Does it follow that $\tan(\tan^{-1}a)=\tan^{-1}(\tan a)$?

18. Prove $\tan^{-1}\{(\sqrt{2}+1)\tan\theta\}-\tan^{-1}\{(\sqrt{2}-1)\tan\theta\}$
$$=\tan^{-1}(\sin 2\theta).$$

19. If $2\tan^{-1}x=\sin^{-1}2y$, find the equation between y and x.

20. Solve $\tan^{-1}(x+1)-\tan^{-1}(x-1)=\cot^{-1}(x^2-1)$.

21. Solve $\sin^{-1}\dfrac{2x}{1+x^2}+\tan^{-1}\dfrac{2x}{1-x^2}=\pi$.

22. Solve the following equations by means of a subsidiary angle:

(1) $\sin\theta+\sqrt{3}\cos\theta=1$.

(2) $\sqrt{3}\sin\theta-\cos\theta=\sqrt{2}$.

(3) $3\cos\theta+4\sin\theta=2.5$.

(4) $3\cos x-8\sin x=3$.

(5) $5\sin x-12\cos x=13$.

(6) $5\cos x+6\sin x=8$.

23. Express the following in factors by means of a subsidiary angle:

(1) $1+\sin A$. (2) $1+a\cos A$. (3) $\dfrac{a\sin A}{1+a\cos A}$.

24. Use a subsidiary angle to determine the sign of the radical in $\cos\dfrac{A}{2}+\sin\dfrac{A}{2}=\sqrt{1+\sin A}$.

25. Trace the changes in the numerical value of $\dfrac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}$ as θ changes from 0° to π.

26. In any triangle prove $a=(b-c)\sec\theta$ if $\tan\theta=\dfrac{2\sqrt{bc}}{b-c}\sin\dfrac{A}{2}$.

27. Explain why $\theta+\dfrac{\pi}{4}=n\pi+(-1)^n\dfrac{\pi}{6}$ and $\theta-\dfrac{\pi}{4}=2n\pi\pm\dfrac{\pi}{3}$, give the same series of angles.

28. Given $3\tan^{-1}\dfrac{1}{4}+\tan^{-1}\dfrac{1}{20}+\tan^{-1}\dfrac{1}{x}=\dfrac{\pi}{4}$, find x.

29. Given $\tan^{-1}\dfrac{1}{a-1}=\tan^{-1}\dfrac{1}{x}+\tan^{-1}\dfrac{1}{a^2-x+1}$, find x.

30. Given $\sec^{-1}a+\sec^{-1}\dfrac{x}{a}=\sec^{-1}b+\sec^{-1}\dfrac{x}{b}$, find x.

31. Given $\tan(\cot\theta)=\cot(\tan\theta)$, find θ.

32. If $\sin(\pi\cos\theta)=\cos(\pi\sin\theta)$, then $\theta=\cos^{-1}\dfrac{1}{2\sqrt{2}}+\dfrac{\pi}{4}$. Verify this value in the given equation and examine the truth of the result $\theta=\pm\dfrac{1}{2}\sin^{-1}\dfrac{3}{4}$.

33. Prove $\tan^{-1}\dfrac{a\cos\phi}{1-a\sin\phi}-\tan^{-1}\dfrac{a-\sin\phi}{\cos\phi}=\phi$.

34. Given $y=\tan^{-1}\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$, prove $x^2=\sin 2y$.

CHAPTER XI.

LOGARITHMS.

155. The logarithm of a number to a given base is the index of the power to which that base must be raised to equal the given number. Thus, if $a^x = N$, then x is the logarithm of N to the base a, and the preceding equation may be written $x = \log_a N$.

Ex.—Find the logarithm of 81 to base 3, and of 1000 to base 10. Since $81 = 3^4$, and $1000 = 10^3$, we have $\log_3 81 = 4$ and $\log_{10} 1000 = 3$.

156. *The logarithm of 1 is 0 for all bases, and the logarithm of the base itself is 1.*

For $a^0 = 1$, therefore by definition $\log_a 1 = 0$.
And $a^1 = a$, therefore $\log_a a = 1$.

In both cases a may have any value whatever; the propositions are, therefore, universally true.

157. It should be carefully remembered that a logarithm is simply an index, or exponent, detached from the base quantity to which it belongs, and made to stand alone on one side of an equation. Thus the three equations,

$$N = a^x, \ x = \log_a N, \ a = \sqrt[x]{N},$$

in which N, x, and a respectively, stand alone, express the same relation of the quantities involved, and consequently any one of

the equations may, at any time, be replaced by either of the other two. Also, replacing x in the first equation by its value from the second equation, we get the important identity, $N = a^{\log_a N}$.

158. The suffix denoting the base is frequently omitted. This may be done when it is perfectly clear what base is understood. It is also omitted in stating general properties which are true whatever base may be employed. Examples of the latter are found in the statement of the Logarithmic Laws, Art. 159, and of the former in their proof where the base a is clearly understood.

159. The Exponential Laws may be written,

I. $a^x \times a^y = a^{x+y}$. II. $a^x \div a^y = a^{x-y}$.

III. $(a^x)^y = a^{xy}$. IV. $(a^x)^{\frac{1}{y}} = a^{\frac{x}{y}}$.

From these we derive the Logarithmic Laws,

I. $\log mn = \log m + \log n$. II. $\log \frac{m}{n} = \log m - \log n$.

III. $\log m^n = n \log m$. IV. $\log \sqrt[n]{m} = \frac{1}{n} \log m$.

These are simply a restatement of the former laws in a different notation, as will appear from the following article.

160. *To prove the Logarithmic Laws.*

Let $m = a^x$, $n = a^y$

then $x = \log_a m$, $y = \log_a n$.

I. $\log mn = \log a^x . a^y = \log a^{x+y} = x + y = \log m + \log n$.

II. $\log \frac{m}{n} = \log \frac{a^x}{a^y} = \log a^{x-y} = x - y = \log m - \log n$.

III. $\log m^n = \log (a^x)^n = \log a^{nx} = nx = n \log m$.

IV. $\log \sqrt[n]{m} = \log (a^x)^{\frac{1}{n}} = \log a^{\frac{x}{n}} = \frac{x}{n} = \frac{1}{n} \log m$.

161. The preceding laws should also be remembered in words as follows:

1. The logarithm of a product is equal to the sum of the logarithms of its factors.

2. The logarithm of a quotient is equal to the logarithm of the dividend, minus the logarithm of the divisor.

3. The logarithm of a power of a number is equal to the logarithm of the number multiplied by the index of the power.

4. The logarithm of the root of a number is equal to the logarithm of the number divided by the index of the root.

We may also observe that by the use of logarithms the operations of multiplication and division are replaced by those of addition and subtraction, whilst involution and evolution are replaced by multiplication and division.

162. The following are applications of the logarithmic laws. No base is expressed, the results being true for all bases.

Ex. 1.—Find the logarithm of 75×48 in terms of the logarithms of 2, 3 and 5.

$$\begin{aligned}\log(75 \times 48) &= \log 75 + \log 48\\ &= \log(5^2 \times 3) + \log(2^4 \times 3)\\ &= \log 5^2 + \log 3 + \log 2^4 + \log 3\\ &= 2\log 5 + 2\log 3 + 4\log 2.\end{aligned}$$

Ex. 2.—Express $\log \dfrac{\sqrt[4]{5} \times \sqrt{20}}{\sqrt[3]{18\sqrt{2}}}$ in terms of log 2, log 3 and log 5.

$$\begin{aligned}\log \frac{\sqrt[4]{5} \times \sqrt{20}}{\sqrt[3]{18\sqrt{2}}} &= \log \sqrt[4]{5} + \log \sqrt{20} - \log \sqrt[3]{18\sqrt{2}}\\ &= \tfrac{1}{4}\log 5 + \tfrac{1}{2}(2\log 2 + \log 5) - \tfrac{1}{3}(2\log 3 + \tfrac{3}{2}\log 2)\\ &= \tfrac{1}{2}\log 2 - \tfrac{2}{3}\log 3 + \tfrac{3}{4}\log 5.\end{aligned}$$

163. Logarithms enable us to solve equations in which the unknown quantity occurs as an exponent.

Ex. 1.—Given $a^x = b$, to find x.

We have $\qquad \log a^x = \log b$

that is $\qquad x \log a = \log b$

or $\qquad x = \dfrac{\log b}{\log a}$.

This result is true whatever may be the base of the logarithms used. If we employ the base a, we have at once by definition, $x = \log_a b$, which agrees with the given result, since $\log_a a = 1$.

Ex. 2.—Given $a^{2x} . b^{1-x} = c^{x+2}$ to find x.

Taking the logarithms of both sides, we get

$$2x \log a + (1-x) \log b = (x+2) \log c$$

or $\qquad x\,(2 \log a - \log b - \log c) = 2 \log c - \log b$

therefore $\qquad x = \dfrac{2 \log c - \log b}{2 \log a - \log b - \log c}$.

164. The logarithm of any given number evidently depends upon the base chosen. If the base be changed, the logarithm must also be changed. We give a few simple examples illustrating the nature of such changes before investigating the theory in its most general form.

Ex. 1.—Since $\qquad 64 = 2^6 = (2^2)^3 = 4^3$

therefore $\qquad \log_2 64 = 6$, and $\log_4 64 = 3$.

From which we observe that when the base is squared the logarithm must be divided by 2.

Ex. 2.—Since $2187 = 3^7 = (3^3)^{\frac{7}{3}} = 27^{\frac{7}{3}}$.

therefore $\qquad \log_3 2187 = 7$, and $\log_{27} 2187 = \dfrac{7}{3}$.

This example shows that when the base is cubed the corresponding logarithm is obtained by dividing the former logarithm by 3.

Ex. 3.—Let $N=a^x$, and let $a^y=b$,

then $$x=\log_a N, \text{ and } y=\log_a b.$$

Then $$N=a^x=(a^y)^{\frac{x}{y}}=b^{\frac{x}{y}}.$$

therefore $$\log_b N=\frac{x}{y}=\frac{\log_a N}{\log_a b}.$$

165. *To find the relation between the logarithms of the same number to different bases.*

Let $\log_a N=x$, it is required to find $\log_b N$.

Let $$y=\log_b N, \text{ so that } b^y=N.$$

Then $$\log_a (b^y)=\log_a N=x,$$

therefore $$y.\log_a b=x,$$

and $$y=\frac{x}{\log_a b},$$

or $$\log_b N=\frac{\log_a N}{\log_a b}.$$

Cor. 1.—For N write a.

Then $$\log_b a=\frac{\log_a a}{\log_a b}=\frac{1}{\log_a b},$$

or $$\log_a b\times\log_b a=1.$$

Cor. 2.—For N write c and simplify.

Then $$\log_a b\times\log_b c=\log_a c.$$

Cor. 3.—In Cor. 2 multiply both sides of the equation by $\log_c a$.

Then $$\log_a b\times\log_b c\times\log_c a=\log_a c\times\log_c a=1.$$

This principle may easily be extended to any number of quantities taken in circular order.

EXERCISE XXII.

1. Express 4, 16, 256, 1024, .125, .03125 in powers of 2, and thus find their logarithms to the base 2.

2. Find the logarithms of 81, 243, $.\dot{3}$, $.\dot{1}$, $\sqrt{27}$ to the base 3.

3. Find the logarithms of 1000, 10, 1, .1, .001 to the base 10.

4. Find the log of 100 to base .001, and of $32\sqrt[5]{4}$ to the base $2\sqrt{2}$.

5. Find the logs of $(25)^{n+1} \times .04$, $\sqrt{5^{n+1} \times 5^{n-1}}$,

$$625\sqrt[3]{25^{-1} \times 125^{-\frac{1}{3}}} \text{ to the base } 5.$$

6. Find the values of

$$\log_8 256,\ \log_7 \frac{\sqrt[3]{7}}{\sqrt{343}},\ \log_4 \sqrt[3]{2} \times \sqrt[5]{8},\ \log_9 \sqrt{3^{2/3}}.$$

7. Express $\log a^2b^3c$, $\log (a^x \div b^y)c^z$, $\log \left\{ \sqrt[3]{a^{-1}\sqrt{b^3}} \div \sqrt{b^3.\sqrt{a}} \right\}$, in terms of $\log a$, $\log b$, and $\log c$.

8. Simplify $\log \dfrac{375}{64} - 2 \log \dfrac{5}{9} + \log \dfrac{64}{243}$.

9. Simplify $\log \dfrac{\sqrt[4]{5}\,.\,\sqrt[10]{12}}{\sqrt[5]{24}\,\sqrt[3]{40}}$, and $\log \sqrt[4]{729\sqrt[3]{9^{-1}\,.\,27^{-\frac{4}{3}}}}$.

10. Given $8\,(2^{x-1})^3 = 4^{x+1}$, find x.

11. Given $\dfrac{4^x}{2^{x+y}} = \dfrac{1}{2}$, $8\sqrt[x]{2} = 4^y$, find x and y.

12 Solve equations

(1) $a^{mx} = b$. (2) $a^x\,.\,b^x = c$. (3) $a^{x+1}.b^{1-x} = c^2\,.\,d^x$.

(4) $2^x = 800$. (5) $(a+b)^2\,(a^2-b^2)^{x-1} = (a-b)^{2x}$. (6) $a^{b^x} = b^{a^x}$.

13. Find two consecutive integers between which the values of the following logarithms lie,

$$\log_3 95,\ \log_5 175,\ \log_7 10,\ \log_4 2,\ \log_5 \tfrac{1}{10},\ \log_{10} .0004.$$

14. Given that N is an integer, and that $\log_5 N > 2$, and that $\log_3 N < 3$; find N.

15. Find the logarithms of the numbers in Ex. 2 to the base 9, and compare them with the logs to the base 3.

16. If $\log_a N = x$, find $\log N$ to the base a^n.

17. If $\log N$ to the base a^n is x, of what number is x the logarithm to the base a?

18. From the identity $N = a^{\log_a N}$, show that $a^{\log b} = b^{\log a}$.

19. If $\log_a N = b$, and $\log_b N = a$, show that

$$b^{\log_a N \,.\, \log_b a} = a^{\log_b N \,.\, \log_a b}.$$

Is this equation true without the given condition?

20. Simplify $a^{\log b} \times a^{\log c}$, $\sqrt{a^{\log b^2}}$, $a^{\log b} \div a^{\log c}$.

21. Show that if a series of numbers are in $G\ P$, their logs are in $A\ P$.

22. Sum the series $\log a + \log ar + \log ar^2 + \ldots$ to n terms.

23. Prove $\log_a m^x . \log_b n^y = \log_a m^y . \log_b n^x = \log_a n^x . \log_b m^y$.

24. Prove $\log_a m^x . \log_b n^y . \log_c r^z = \log_b r^x . \log_c m^y \log_a n^z$
$= \log_b m^y . \log_c n^z . \log_a r^x$.

25. Eliminate x from the equations $a^x = m$, $b^x = n$.

26. If $x = \log_a m = \log_b n$, then $x = \log_{ab} mn = \log_{\frac{a}{b}} \dfrac{m}{n}$.

27. If $x = \log_a m = \log_b n = \log_c p$, then $x = \log m^r n^s p^t$ to the base $a^r b^s c^t$.

28. Prove $2 \log \sin A = \log (1 + \cos A) + \log (1 - \cos A)$.

29. Prove $\log \sin 2A = \log 2 + \log \sin A + \log \cos A$.

30. Prove $\log \cos 2A = \log (\cos A + \sin A) + \log (\cos A - \sin A)$.

31. Prove $\log \sin A - \log \cos A = \log \tan A$.

32. Find the value of $\log \tan 45°$, and $\log (\sec A + \tan A) + \log (\sec A - \tan A)$.

33. Simplify $\log \tan A - \log (1 + \tan A) - \log (1 - \tan A) + \log 2$.

Different Systems of Logarithms.

166. Any positive number different from unity may be chosen as the base of a system of logarithms, but only two different systems are in common use. The first system, called **Napierian logarithms,** from their discoverer, Baron Napier, has for its base the sum of the series

$$1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}\ldots\text{ad. inf.}$$

This number is incommensurable, and consequently it can be only approximately expressed. Its value to eight decimal places is 2.718281828, and is usually denoted by e. They are also called **natural logarithms,** because logarithms to that base are the most easily calculated. They are always used in theoretical investigations. The second system, called **common logarithms,** has for its base the number 10, the radix of our system of notation, and is always used for practical work. The great advantage of this choice of base will be easily perceived from Art. 171.

167. Writing the successive powers of 10, both positive and negative, $10^0=1$ $\therefore$ $\log 1=0$.

$$\begin{array}{llll}
10^1=10, & \therefore \log 10=1. & 10^{-1}=.1, & \therefore \log .1=-1. \\
10^2=100, & \therefore \log 100=2. & 10^{-2}=.01, & \therefore \log .01=-2. \\
10^3=1000, & \therefore \log 1000=3. & 10^{-3}=.001, & \therefore \log .001=-3.
\end{array}$$

From the above we observe:

1. The logarithms of numbers greater than unity are *positive.*
2. The logarithms of numbers less than unity are *negative.*
3. Negative numbers have no real logarithms.
4. The logarithms of any number between two consecutive powers of 10 will be between the consecutive integers denoting those powers, *i.e.*, it will be a whole number and a fraction (or decimal).

It will be readily perceived that whatever base may be chosen, but comparatively few numbers will have exact logarithms. Thus in the common system only the numbers 10, 100, 1000, etc., have exact logarithms. The logarithms of all other numbers are incommensurable. For example, the common logarithm of 67 is the value of x, for which $10^x = 67$, but this value can be only approximately obtained. Its value to 5 decimal places is 1.82607, which means that $10^{1.82607} = 67$, or that $10^{182607} = 67^{100000}$ approximately. To verify this equality by actual multiplication would take more time than the student is likely to have at his disposal.

168. In dealing with negative logarithms it is most convenient in practice to express them in a form in which the integral part alone is negative. In such cases the negative sign is written *over* the integral part; thus $\bar{2}.75842$ means $-2 + .75842$.

169. When a logarithm is expressed with its decimal part positive the integral part is called the **characteristic,** and the decimal part the **mantissa.**

Ex. 1.—The number 275 is between 10^2 and 10^3, therefore log 275 is between 2 and 3. The value is found to be 2.43933, in which 2 is the characteristic and .43933 is the mantissa.

Ex. 2.—The number .0275 is between 10^{-2} and 10^{-1}, therefore its log is between -2 and -1; its value is found to be $-(1.56067)$.

In this case, however, the characteristic is not -1, nor is the mantissa .56067, for the decimal part is *negative.* Transferring the log to the proper form, we have

$$\begin{aligned} -1.56067 = -1 - .56067 &= -2 + 1 - .56067 \\ &= \bar{2}.43933, \end{aligned}$$

from which we see that -2 is the characteristic and .43933 the mantissa.

170. We give a few simple examples to show the proper method of dealing with logarithms having negative characteristics:

(1)	(2)	(3)
3.14682	$\bar{1}$.24638	3 | $\bar{2}$.48256
$\bar{2}$.56347	5	$\bar{1}$.49419
4.58335	$\bar{4}$.23190	

In Ex. (1) the subtraction is performed in the ordinary way until we reach the negative characteristic $\bar{2}$. There is 1 to carry from the preceding column which added to $\bar{2}$ makes $\bar{1}$, and this subtracted algebraically from 3 gives 4.

In Ex. (2), multiplying in the ordinary way there is 1 to carry from the decimal to the characteristic; this added to 5 times $\bar{1}$ gives $\bar{4}$.

In Ex. (3) we add $\bar{1}$ to the characteristic, and $+1$ to the decimal; this does not alter the value of the given logarithm, and it renders the integral part exactly divisible by the given divisor.

171. *To find the characteristic of the logarithm of any number greater than unity.*

Let N denote the given number, and let it contain n digits in the integral part.

Since 10^{n-1} is the smallest number containing n digits,

therefore $$10^{n-1},\ N,\ 10^n,$$

are in ascending order of magnitude, and consequently $\log N$ lies between $n-1$ and n.

Therefore $$\log N = (n-1) + \text{a decimal},$$

or $n-1$ is the characteristic of $\log N$.

The characteristic of the logarithm of a number greater than unity is positive, and less by unity than the number of digits in the integral part.

172. *To find the characteristic of the logarithm of a decimal fraction.*

Let N denote the given number, and let n denote the number of zeros between the decimal part and the first significant figure.

Since $10^{-(n+1)}$ is the least number containing only n zeros between the decimal part and the first significant figure,

therefore $$10^{-(n+1)}\ N,\ 10^{-n}$$

are in ascending order of magnitude, and consequently $\log N$ lies betweeen $-(n+1)$ and $-n$.

Therefore $$\log N = -(n+1) + \text{a decimal},$$

or $-(n+1)$ is the characteristic of $\log N$.

The characteristic of the logarithm of a decimal fraction is negative, and numerically greater by unity than the number of zeros between the decimal part and the first significant figure.

173. *If two numbers differ only in the position of the decimal point, their logarithms differ only in their characteristics.*

Let N and N' denote two such numbers of which N is the greater, and let r denote the difference in the number of their decimal places, then

$$N = N' \times \text{by } 10^r$$

and $$\log N = \log N' + \log 10^r \qquad \text{Art. 161.}$$
$$= \log N' + r.$$

Now r is an integer, and consequently its addition to $\log N'$ will change only the characteristic.

Examples.—The characteristics of the logarithms of 37, 5, 87625.43 are 1, 0, 4 respectively, these numbers being each less by a unit than the number of digits in the integral part of the corresponding number.

The characteristics of the logarithms of .06, .357, .000347 are -2, -1, -4 respectively, these numbers being each negative and numerically greater by a unit than the number of zeros between the decimal point and the first significant figure.

Given	log	$3782576 = 6.5777877$
then	log	$37.82576 = 1.5777877$
	log	$.003782576 = 3.5777877$, etc.

The advantages of the common system of logarithms are now evident They are:

1. The characteristic can be written at once by inspection, and consequently need not be registered in the tables.

2. The mantissæ are the same for all numbers consisting of the same digits in the same order, and consequently the mantissæ of integers only need be given.

The explanation of the method by which tables of logarithms are constructed does not lie within the scope of this work, but the student may be informed that logarithms are first calculated to the base e, and then transformed into common logarithms by the principle of Art. 165.

Thus
$$\log_{10} N = \frac{\log_e N}{\log_e 10} = \log_e N \times \frac{1}{\log_e 10},$$

so that when a table of Napierian logarithms has been formed it may be changed into a table of common logarithms by multiplying each by the constant factor,

$$\frac{1}{\log_e 10} = \frac{1}{2.30258509\,..} = .43429448\ldots.$$

which is called the **modulus** of the common system of logarithms.

174. In the Appendix is given a table of the logarithms of all integral numbers from 100 to 1000. As just explained, only the mantissæ are registered, and since these are all decimals, the decimal point is omitted. Also when only the last three figures

of the logarithm are given, the first two are to be supplied from the number next preceding which has the logarithm given in full.

From the table given, the logarithm of any number consisting of not more than five digits can be obtained by methods which we shall now explain by means of examples. We again assume the principle of proportional parts, Art. 77.

Ex. 1.—Find log 385, log .0385, and log 38500.

In the tables opposite 385 we find 58546, which is the mantissa for each of the numbers, since they all consist of the same digits. Their characteristics are 2, -2, and 4 respectively.

Hence
$$\log 385 = 2.58546$$
$$\log .0385 = \bar{2}.58546$$
$$\log 38500 = 4.58546.$$

Ex. 2.—Find log 28763.

From the tables we have
$$\log 28800 = 4.45939$$
$$\log 28700 = 4.45788$$

from which $\quad$ difference for $100 = 151$

therefore $\quad$ difference for $63 = 151 \times .63$
$$= 95.13.$$

Then to $\quad \log 28700 = 4.45788$

add $\quad$ difference for $63 = 95$

therefore $\quad \log 28763 = 4.45883.$

Ex. 3.—Find the number whose logarithm is 2.34698.

From the tables we find,

log 223 = 34830	given log = 34698
log 222 = 34635	log 222 = 34635
diff. for 1 = 195	given diff. = 63

Then $\frac{63}{195} = .32$, which added to 222 gives 222.32, the number required.

Explanation.—In dealing with the mantissa we take no notice of the characteristic. Since the same number of decimal places is employed in each case, the decimal point is omitted. And since an increase of 195 in the logarithm gives an increase of 1 in the number, an increase of 63 in the logarithm gives an increase of $\frac{63}{195}$, .32 in the number. Finally, we observe that the given characteristic is 2, and consequently there must be three significant figures in the integral part of the number.

Had the given characteristic been different the position of the decimal point in the final result would have been different; otherwise the work would have been exactly the same. For example, $\bar{2}.34698$ is the logarithm of .022232. The mantissa 34698 depends upon the sequence of digits 22232; the characteristic, $\bar{2}$, gives the position of the decimal point.

175. Logarithms are extensively used for simplifying expressions requiring tedious multiplications, divisions, or the extraction of roots. The general plan is to find the logarithm of the given expression, and then find the number corresponding to this logarithm.

Ex.—Simplify $\dfrac{.0084321 \times (\frac{2}{15})^{\frac{1}{3}}}{\sqrt{8.37}}$.

$$\begin{aligned} \log 84400 &= 92634 \\ \log 84300 &= 92583 \\ \text{diff. for } 100 &= 51. \end{aligned} \qquad \begin{aligned} \text{diff. for } 21 &= \tfrac{21}{100} \text{ of } 51 \\ &= 11 \text{ nearly.} \end{aligned}$$

$$\begin{array}{lrr} \text{Then to} & \log 84300 = & 92583 \\ \text{add} & \text{diff. for } 21 = & 11 \\ \text{therefore} & \log .0084321 = & \bar{3}.92594. \end{array}$$

$$\log (\tfrac{2}{15})^{\frac{1}{3}} = \tfrac{1}{3}(\log 2 - \log 15) = \bar{1}.70831.$$

$$\log \sqrt{8.37} = \tfrac{1}{2} \log 8.37 \qquad = .46136.$$

Therefore log of given fraction

$$= \bar{3}.92594 + \bar{1}.70831 - .46136$$
$$= \bar{3}.17289$$

$$\begin{array}{lrr}
\text{Then} & \log 149 = 17319 & \text{given } \log = \dot{1}7289 \\
 & \log 148 = \underline{17026} & \log 148 = \underline{17026} \\
 & 293 & 293\ )\ 26300\ (\ 89 \\
 & & \underline{2344} \\
 & & 2860 \\
 & & \underline{2637}
\end{array}$$

Therefore $\bar{3}.17289 = \log .0014889$.
The given fraction $= .0014889$, which is the result required.

EXERCISE XXIII.

1. Add $\bar{2}.07895$, 3.67893, $\bar{5}.34785$, $.89207$.

2. From 1.07638 take 4.25763, and from $\bar{3}.48273$ take $\bar{1}.38405$.

3. From $.26875$ take $\bar{2}.39607$, and from the result take 1 82753.

4. Multiply 2.37654 by -5, and $\bar{3}.20763$ by 4.

5. Divide $\bar{2}.34687$ by 3, and by -4.

6. Divide $\bar{1}.34067$ by 2.56834, and by $\bar{2}.68235$.

7. Find by inspection the characteristics of the logarithms of

$$1827.54,\ 4.07,\ .0003,\ \frac{1}{125},\ \sqrt[10]{.3765},\ \sqrt[5]{82763}.$$

8. The mantissa of log 8576 is .9332848, write down the logs of 8.576, 857600, .008576.

9. From the preceding example write down the numbers whose logs are 2.9332848, $\bar{2}.9332848$, 7.9332848.

10. How many positive integers are there whose logarithms to base 3 have 5 for characteristic?

11. Find from the tables the logarithms of the following:

(1) 28507. (2) 3.8762. (3) .0623. (4) .0075.
(5) $(1.05)^7$. (6) $\sqrt{3.1416}$. (7) $(1.06)^{-\frac{1}{4}}$. (8) $(.002)^{-\frac{1}{3}}$.

12. Find from the tables the numbers of which the following are the logarithms:

(1) 1.14613. (2) $\bar{1}.24650$. (3) 4.18037. (4) $\bar{1}.00042$.

13. Write down the logarithms of the numbers in Ex. 8 to the base 100 and to the base .01.

14. How many digits are there in the integral part of $(1.05)^{2000}$?

15. Given that the integral part of $(3.456)^{100000}$ contains 53856 digits, find log 345.6 correct to five places of decimals.

16. Given log $2 = .3010300$, find the logarithms of

$$8,\ \tfrac{1}{2},\ 5,\ 800,\ \sqrt[3]{2},\ .03125,\ \frac{1}{\sqrt[5]{128}},\ \sqrt{2} \times \sqrt[3]{5}.$$

17. Given log $3 = .4771213$, find the logarithms of

$$30,\ \sqrt{90},\ .3,\ \dot{3},\ .0\dot{1}\ \ \sqrt[5]{24.3},\ (.081)^{-\frac{1}{3}}.$$

18. From the known values of log 2 and log 3, find the logarithms of $2\frac{2}{5}$, 1.44, $1.\dot{8}5\dot{1}$, $(1\frac{2}{3})^{20}$, $\sqrt[5]{\dfrac{3^{\frac{1}{4}}.5^{\frac{3}{4}}}{750}}$.

19. From log 2, log 3, and log $7 = .8450980$, find the logs of

$$\sqrt[3]{28} \times \sqrt[4]{7.5},\ \sqrt{.56} \div \sqrt[5]{.0049},\ (.0126)^{\frac{3}{4}} \times (3.43)^{-\frac{1}{5}} \div (.044\dot{1})^{\frac{1}{4}}.$$

20. Simplify

$$\frac{1.28}{1.25} \times \frac{(216)^{\frac{3}{5}}}{.81} \times \frac{5}{\sqrt[5]{1.2}},\ \text{given } \log 15193.64 = 4.1816618.$$

21. Which is the greater, $(1\frac{1}{3})^{15}$, or $(1\frac{1}{2})^{10}$, given log 2 and log 3?

22. Find by logarithms the cube root of 7.

23. Find by logarithms the fifth root of 27.65.

24. Simplify $\sqrt[4]{80} \times \sqrt[3]{2.7} \times \sqrt[5]{-5} \times 18^{-\frac{1}{5}}$.

25. Solve the following equations to three places of decimals:

(1) $5^x = 300$. (2) $8^x = 5\,(3)^{x+2}$. (3) $(\frac{4}{7})^x = 54\frac{1}{2}$.
(4) $2^{3x} \times 3^{2x} = 4.9$. (5) $15^x = (2.5)^{1-x}$. (6) $3^{5x} = 1000$.

26 In how many years will any sum of money double itself at $3\frac{1}{2}$ per cent.?

27. In how many years will any sum of money amount to 10 times as much at 6 per cent. as at 5 per cent.?

28. Find the edge of a cube whose volume equals that of the earth, supposing the latter to be a sphere whose diameter is 7912 miles.

29. Solve equation

$\frac{x}{2}$ $(\frac{1}{8})^x\,(125)^{1-x} = (\frac{1}{4})^{3x+2}\,(\frac{1}{5})^x$ to three decimal places.

30. Given $2^x = 7y$, $3^x = 10y$, find x and y.

31. Given $\frac{3^x}{2^y} = 4$, $7^x = 3^y$, find x and y.

32. Given log 1.44 = .1583625, log 16.2 = 1.2095150, find log 2 and log 3.

33. Find $\log_9 270$ and $\log_5 10$ from the known values of log 3 and log 2.

34. Given $\log_e 2 = .693147$, $\log_e 5 = 1.609438$, find the common log of 6.4.

35. Find $\log_e 7$ from the known values of $\log_{10} 7$, $\log_e 2$, $\log_e 5$, and thence find $\log_2 7$.

36. Given $\log_e 3 = 1.098612$, $\log_e 6 = 1.791759$, find $\log_2 12$.

37. Find approximately the value of $5^{\sqrt{5}}$.

38. Find the value of $\frac{.2 \times .4 \times .8 \ldots \text{ to 12 factors}}{.3 \times .9 \times 2.7 \ldots \text{ to 9 factors}}$.

CHAPTER XII.

SOLUTION OF TRIANGLES BY LOGARITHMS.

176. In Chapter V. we deduced the most important relations between the sides and angles of a triangle, and gave examples of their application. We shall now show how logarithms are employed to abbreviate the labor when numbers consisting of several digits are involved.

177. The sines, cosines and tangents of all angles between 0° and 45° are each less than unity and consequently their logarithms are negative. In practice it is found convenient to render all such logarithms positive by adding 10 to each. The logarithms of the trigonometrical ratios thus increased by 10 are called Tabular Logarithms, and are distinguished from true logarithms by the letter L. Thus, L sin 15° means log sin $15° + 10$. The logarithmic cotangents being already positive, the 10 may be omitted and their true value given.

178. For convenience of reference we have given in the Appendix the tabular logarithms of the ratios most frequently employed for angles between 0° and 45° at intervals of 10′. For angles of intermediate value the principle of proportional parts is employed. For angles greater than 45° the proper ratios of the complementary or supplementary angles must be taken. It should be observed, however, that for angles greater than 90°, the tan, cot, cos and sec are all negative, and consequently they have no real logarithms. The values of ratios not given, viz., the secant and cosecant, may easily be deduced from those of the cosine and sine by methods which we shall now explain.

179. *The tabular logarithm of any ratio is found by subtracting that of its reciprocal from 20.*

We have
$$\begin{aligned} L \operatorname{cosec} A &= 10 + \log \operatorname{cosec} A \\ &= 10 + \log \frac{1}{\sin A} \\ &= 10 - \log \sin A \\ &= 10 - (L \sin A - 10) \\ &= 20 - L \sin A. \end{aligned}$$

Similarly $\qquad L \sec A = 20 - L \cos A$

and $\qquad L \cot A = 20 - L \tan A.$

Ex. 1—Find the value of $L \sin 25° 13' 20''$.

From the tables we find:

$$\begin{array}{ll} L \sin 25° 20' & = 9.63133 \\ L \sin 25° 10' & = 9.62865 \\ \hline \text{Diff. for } 10' & \quad 268 \\ \text{Diff. for } 3' 20'' = & 26.8 \times 3\tfrac{20}{60} \\ = & \quad 89. \end{array} \qquad \begin{array}{lr} \text{Then to } L \sin 25° 10' = & 9.62865 \\ \text{add diff. for } 3' 20'' = & 89 \\ \hline \text{Result} & = 9.62954. \end{array}$$

Ex. 2.—Find the value of $L \cos 22° 37' 45''$.

From the tables we find:

$$\begin{array}{ll} L \cos 22° 30' & = 9.96562 \\ L \cos 22° 40' & = 9.96509 \\ \hline \text{Diff. for } 10' = & \quad 53 \\ \text{Diff. for } 7' 45'' = & 5.3 \times 7\tfrac{45}{60} \\ = & \quad 41. \end{array} \qquad \begin{array}{lr} \text{Then from } \cos 22° 30' = & 9.96562 \\ \text{take diff. for } 7' 45'' = & 41 \\ \hline \text{Result} & = 9.96521. \end{array}$$

Ex. 3.—Find the angle whose logarithmic sine is 9.80537.

From the tables we find:

$$\begin{array}{ll} L \sin 39° 50' = & 9.80656 \\ L \sin 39° 40' = & 9.80504 \\ \hline \text{Diff. for } 10' = & \quad 152. \end{array} \qquad \begin{array}{lr} \text{Given log} & = 9.80537 \\ L \sin 39° 40' = & 9.80504 \\ \hline \text{Given diff.} & = \quad 33. \end{array}$$

Then $\frac{33}{152}$ of $10' = 2' 10''$; and $39° 40' + 2' 10'' = 39° 42' 10''$ is the angle required.

Ex. 4.—Find the angle whose logarithmic cosine is 9.87900.

From the tables we find:

$L \cos 40^\circ\ 40' =$	9.87996	$L \cos 40^\circ\ 40' =$	9.87996
$L \cos 40^\circ\ 50' =$	9.87887	Given log $=$	9.87900
Diff. for $10' =$	$109.$	Given diff. $=$	$96.$

Then $\frac{96}{109}$ of $10' = 8'\ 48''$, and $40^\circ\ 40' + 8'\ 48'' = 40^\circ\ 48'\ 48''$ is the angle required.

180. When an angle is to be found from its logarithmic sine, should the given value be greater than $L \sin 45^\circ$, we must find the angle whose cosine has the given value. The difference between this angle and 90° will evidently be the angle required. An example is found in Art. 194, where B is to be found from $L \sin B = 9.98556$. We find $9.98556 = L \cos 14^\circ\ 41'\ 31''$, then $90^\circ - 14^\circ\ 41'\ 31'' = 75^\circ\ 18'\ 29''$, the angle required. The same method may evidently be pursued with regard to any two complementary ratios.

181. When a given logarithmic tangent is greater than $L \tan 45^\circ$, *i.e.*, greater than 10, the method of the last article may be employed, but it is more convenient to reason thus:

$$L \tan (90^\circ - A) = L \cot A = 20 - L \tan A.$$

The value of $L \tan (90^\circ - A)$ will be found in the tables from which $90^\circ - A$ is found, and then A is known. This method is employed in Art. 186, to find B from its logarithmic tangent, which is greater than 10, and consequently is not found in the table of tangents. The same method evidently applies to any ratio and its reciprocal.

Right-Angled Triangles.

182. A triangle contains six elements, three sides and three angles. When three elements are given, one of them being a side, the others can usually be found. We shall begin with

right-angled triangles and assume the notation of Art. 72. Since one angle is always known, being a right angle, but two other elements need be given. We shall have four cases to consider as follows:

183. *Case I.*—Given the **hypothenuse** and an **acute angle,** $c = 753.2$, $A = 38^\circ\ 25'$.

We have $a = c \sin A$, Art. 72.

therefore
$$\begin{aligned} \log a &= \log c + \log \sin A \\ &= \log 753.2 + L \sin 38^\circ . 25' - 10 \\ &= 2.87691 + 9.79335 - 10 \\ &= 2.67026 \end{aligned}$$

from which $a = 468.01$.

Again $b = c \cos A$, Art. 72.

therefore
$$\begin{aligned} \log b &= \log c + \log \cos A \\ &= \log 753.2 + L \cos A - 10 \\ &= 2.87691 + 9.89405 - 10 \\ &= 2.77096 \end{aligned}$$

from which $b = 590.14$.

Also $B = 90^\circ - A = 51^\circ . 35'$.

184. *Case II.*—Given the **hypothenuse** and a **side,** $c = 753.2$, $a = 468.01$.

We have $\sin A = \frac{a}{c}$. Art. 72.

Therefore
$$\begin{aligned} L \sin A &= \log a - \log c + 10 \\ &= \log 468.01 - \log 753.2 \\ &= 2.67026 - 2.87691 + 10 \\ &= 9.79335 \end{aligned}$$

from which $A = 38^\circ\ 25'$.

B and b can now be found as in Case I. We can also find b without finding either of the angles.

We have $\quad b^2 = c^2 - a^2$ $\qquad$ Euc. I., 47.

$$= (c+a)(c-a)$$

therefore $\quad 2 \log b = \log (c+a) + \log (c-a)$

$$= \log 1221.21 + \log 285.19$$

or $\quad 2 \log b = 3.08679 + 2.45413$

$$\log b = 2.77096$$

and $\quad b = 590.14.$

185. *Case III.*—Given a **side** and an **acute angle.**

$$a = 4596, \; A = 68° \; 17' \; 30''$$

We have $\quad c = \dfrac{a}{\sin A}.$ $\qquad$ Art. 72.

Therefore $\quad \log c = \log a - \log \sin A$

$$= \log 4596 - L \sin 68° \; 17' \; 30'' + 10$$

$$= 3.66238 - 9.96805 + 10$$

$$= 3.69433$$

from which $\quad c = 2341.2.$

B and b can now be found as before.

186. *Case IV.*—Given the **two sides,** $a = 345.8$, $b = 217.6$.

We have $\quad \tan A = \dfrac{a}{b}.$ $\qquad$ Art. 72.

Therefore $\quad L \tan A = \log a - \log b + 10$

$$= \log 345.8 - \log 217.6 + 10$$

$$= 2.53882 - 2.33766 + 10$$

$$= 10.20116.$$

Then $\quad L \tan (90° - A) = 9.79884$

from which $\quad 90° - A = 32° \; 0' \; 51''$

or $\quad A = 57° \; 59' \; 9''.$

B and c can now be found as before. We might find c from the relation $c^2 = a^2 + b^2$, but this is not adapted to the use of logarithms.

EXERCISE XXIV.

1. Find from the tables the value of the following:

(1) $L \sin 23^\circ 15' 40''$. (2) $L \cos 22^\circ 14' 35''$.
(3) $L \tan 37^\circ 25' 52''$. (4) $L \cot 17^\circ 49' 34''$.
(5) $L \sec 32^\circ 19' 23''$. (6) $L \operatorname{cosec} 18^\circ 35' 17''$.

2. Find the angle A in the following equations:

(1) $L \sin A = 9.42376$. (2) $L \cos A = 9.85623$.
(3) $L \sin A = 9.97628$. (4) $L \tan A = 10.34598$.
(5) $L \sec A = 10.07293$. (6) $L \cot A = 10.18942$.

3. Given $L \tan 32^\circ 32' = 9.8047447$, diff. for $1' = 2786$, find $L \tan 32^\circ 32' 32''$, and $L \cot 57^\circ 27' 14''$.

4. Given $L \cos 21^\circ 21' = 9.9691241$, diff. for $1' = 495$, find $L \cos 21^\circ 21' 21''$, and $L \sin 68^\circ 38' 25''$.

5. From Ex. 3, find the angles whose L tan and L cot are 9.8047983 and 9.8048235 respectively.

6. From Ex. 4, find the angle whose L sin and L cos are 9.9691037 and 9.9690884 respectively.

7. Find from the tables the value of $L \tan 38^\circ 25' 20''$, and verify the result by finding the logarithm of the natural tangent.

8. Prove that $L \tan A = L \sin A - L \cos A + 10$.

9. Given $L \tan 15^\circ 20' = 9.4380587$, diff. for $1' = 4951$, $L \cos 15^\circ 20' = 9.9842589$, diff. for $1' = 347$; find $L \sin 15^\circ 20' 35''$, $L \sec 15^\circ 20' 41''$, $L \sin 74^\circ 39' 18''$.

10. Given $L \sin 10^\circ 15' = 9.2502822$, diff. for $1' = 6981$, find the angle whose L cos is 9.2503940.

11. Show how to find the value of $L \sin 2A$ from the known values of $L \sin A$ and $L \cos A$. Could the value of $L \cos 2A$ be found in the same way?

12. Show from an examination of the tabular logarithms that the sines and tangents of very small angles are proportional to the angles themselves.

13. Solve the triangle ABC when the following parts are given, C denoting a right angle:

(1) $c = 432$, $A = 18° \ 14'$.	(2) $c = 1234$, $B = 25° \ 19' \ 13''$.
(3) $c = 957.34$, $b = 240$.	(4) $c = 598.2$, $b = 501.8$.
(5) $A = 35° \ 15''$, $a = 86.34$,	(6) $A = 72° \ 15' \ 20''$, $b = 365$.
(7) $a = 75.384$, $b = 14.826$.	(8) $a = 391.4$, $b = 89.62$.

14. In a triangle right-angled at C, $a = 250$, $b = 753$; find the perpendicular from C on the opposite side.

15. In a right-angled triangle, the acute angles are proportional to 2 and 3 and the area is a quarter of an acre; find the sides in rods.

16. At 120 feet from the foot of a steeple the angle of elevation of the top was found to be $60° \ 30'$; find its height.

17. From the top of a perpendicular rock 326 feet high the angle of depression of a point on a horizontal plane below is $24°$; find the distance of the point from the bottom of the rock.

18. Two observers on the same side of a balloon, in the same vertical plane with it, and a mile apart, find its angles of elevation to be $15°$ and $65° \ 30'$; find the height of the balloon.

19. From the bank of a river the top of a tower on the opposite bank is at an elevation of $54°$; 35 feet from the bank its elevation is $49°$; find the breadth of the river, the points of observation and the tower being in a line perpendicular to the river.

20. The angle of elevation of a hill from a point due north of it is $53° \ 18' \ 27''$, and from another point due west of the former and distant from it 430.31 feet, the elevation is $49° \ 17' \ 18''$; find the height of the hill.

Oblique-Angled Triangles.

187. We now give the solution of oblique-angled triangles. The methods employed are applicable to all triangles whatsoever, but those already given are simpler when one of the angles is known to be a right angle. There are four distinct cases which we shall discuss in order.

188. *Case I.*—Given the **three sides,** a, b, c. Euc. I., 8.

From Art. 83 (3) $$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Then $$L\tan\frac{A}{2}=\frac{1}{2}\{\log(s-b)+\log(s-c)-\log s-\log(s-a)\}+10.$$

The value of $L\tan\frac{A}{2}$ being thus known, $\frac{A}{2}$ may be found from the tables. Similarly $\frac{B}{2}$ and $\frac{C}{2}$ may be found. Having thus found the three angles, the accuracy of the work may be tested by adding them and comparing their sum with the proper value, 180°.

Ex.—Given $a=3725$, $b=4873$, $c=6258$, find A, B, C.

In this case $s=7428$, $s-a=3703$, $s-b=2555$, $s-c=1170$.

Then
$$\begin{aligned}L\tan\frac{A}{2}&=\frac{1}{2}\{\log 2555+\log 1170-\log 7428-\log 3703\}+10\\&=\tfrac{1}{2}\{3.40739+3.06819-3.87087-3.56855\}+10\\&=9.51808,\end{aligned}$$

from which $\frac{A}{2}=18°\ 14'\ 45''$, or $A=36°\ 29'\ 30''$.

Similarly $L \tan \dfrac{B}{2} = \dfrac{1}{2}\{\log(s-c) + \log(s-a) - \log s - \log(s-b)\}$
$+10$

$= \dfrac{1}{2}\{3.06819 + 3.56855 - 3.87087 - 3.40739\}$
$+10$

$= 9.67924,$

from which $\dfrac{B}{2} = 25° \ 32' \ 17''$, or $B = 51° \ 4' \ 34''$.

Again $L \tan \dfrac{C}{2} = \dfrac{1}{2}\{3.56855 + 3.40739 - 3.87087 - 3.06819\}$
$+10$

$= 10.01844$

Therefore $L \tan\left(90° - \dfrac{C}{2}\right) = 20 - 10.01844$ Art. 181.

$= 9.98156,$

from which $90° - \dfrac{C}{2} = 43° \ 47' \ 2''$, or $C = 92° \ 25' \ 56''$.

The sum of the three angles thus found is 180°, showing the work to be correct.

189. In the preceding article we might also employ the formula for $\sin \dfrac{A}{2}$, or $\cos \dfrac{A}{2}$, and if we were required to find but one angle either of these would serve our purpose equally well. But the formula for $\tan \dfrac{A}{2}$ requires only four logarithms to find all the angles, whilst either of the others requires six. The formula for $\cot \dfrac{A}{2}$, or for $\sin A$, might also be employed, though the latter would require more logarithms. By using a table of natural cosines the formula for $\cos A$, Art. 82 (2), might be employed; but since it does not consist of factors, it is not adapted for use with logarithms.

190. *Case II.*—Given **one side** and **two angles**, c, A, B,
Euc. I., 26.

Since $A+B+C=180°$, we have at once $C=180°-(A+B)$.

Then from Art. 80, $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$,

from which $a=\dfrac{c \sin A}{\sin C}$, $b=\dfrac{c \sin B}{\sin C}$.

Then
$$\log a=\log c+L \sin A-L \sin C$$
$$\log b=\log c+L \sin B-L \sin C.$$

From these equations a and b may be found.

Ex.—Given $c=338.65$, $A=53° 24'$, $B=66° 27'$; find C and a.

We have $C=180°-(53° 24'+66° 27')=60° 9'$,

Then
$$\log a=\log 338.65+L \sin 53° 24'-L \sin 60° 9',$$
$$=2.52975+9.90462-9.93819,$$
$$=2.49618,$$

from which $a=313.46$.

191. *Case III.*—Given **two sides** and the **included angle**, a, b, C.
Euc. I., 4.

We have $\dfrac{A+B}{2}=90°-\dfrac{C}{2}$ so that $\dfrac{A+B}{2}$ is known.

From Art. 85, $\tan \dfrac{A-B}{2}=\dfrac{a-b}{a+b} \cot \dfrac{C}{2}$.

Then
$$L \tan \frac{A-B}{2}=\log (a-b)-\log (a+b)+L \cot \frac{C}{2}.$$

From this equation $\dfrac{A-B}{2}$ may be found. Then from the known values of $\dfrac{A-B}{2}$ and $\dfrac{A+B}{2}$, the values of A and B may be found. Then to find c we can proceed as in Case II.

Ex.—Given $a=723$, $b=259$, $C=35°\ 18'$; find A, B, c.

We have $\quad a-b=464,\ a+b=982,\ \dfrac{C}{2}=17°\ 39',$

$$\text{Then}\quad L\tan\frac{A-B}{2}=\log 464-\log 982+L\cot 17°\ 39'$$
$$=2.66652-2.99211+10.49733$$
$$=10.17174$$

from which $\quad \dfrac{A-B}{2}=56°\ 2'\ 40''$

and $\quad \dfrac{A+B}{2}=72°\ 21',$

therefore $\quad A=128°\ 23'\ 40'',\ B=16°\ 18'\ 20''.$

Then $\quad c=\dfrac{b\sin C}{\sin B},$

$$\text{or}\quad \log c=\log b+L\sin C-L\sin B$$
$$=\log 259+L\sin 35°\ 18'-L\sin 16°\ 18'\ 20''$$
$$=2.41330+9.76182-9.44834$$
$$=2.72678,$$

from which $\quad c=533.06.$

192. The formula used in the preceding article was proved geometrically in Art. 85; we now give a symbolical proof of it and two similar formulæ which are sometimes useful.

From $\quad \dfrac{a}{\sin A}=\dfrac{b}{\sin B},$ we get $\dfrac{a}{b}=\dfrac{\sin A}{\sin B},$

$$\text{Then}\quad \frac{a-b}{a+b}=\frac{\sin A-\sin B}{\sin A+\sin B}=\frac{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}$$
$$=\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\frac{\tan\frac{1}{2}(A-B)}{\cot\frac{1}{2}C},$$

$$\text{or}\quad \tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}. \qquad (1)$$

Again from $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ we get $\frac{c}{\sin C} = \frac{a+b}{\sin A + \sin B}$.

Then

$$\frac{c}{a+b} = \frac{\sin C}{\sin A + \sin B} = \frac{2 \sin \frac{1}{2} C \cos \frac{1}{2} C}{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)} = \frac{\sin \frac{1}{2} C}{\cos \frac{1}{2}(A-B)},$$

from which $$c = \frac{(a+b) \sin \frac{1}{2} C}{\cos \frac{1}{2}(A-B)}. \tag{2}$$

Similarly we can show $$c = \frac{(a-b) \cos \frac{1}{2} C}{\sin \frac{1}{2}(A-B)}. \tag{3}$$

We may observe that from any two of these three formulæ the remaining one may be obtained. We shall use (2) to find c in the example of the preceding article.

We have
$$\begin{aligned} \log c &= \log(a+b) + L \sin \tfrac{1}{2} C - L \cos \tfrac{1}{2}(A-B) \\ &= \log 982 + L \sin 17^\circ\, 39' - L \cos 56^\circ\, 2'\, 40'' \\ &= 2.99211 + 9.48173 - 9.74706 \\ &= 2.72678, \end{aligned}$$

from which $c = 533.06$ as before.

This method has the advantage of requiring only two new logarithms, whilst the former required three.

193. When two sides and the included angle are given, the third side is given at once by the formula

$$c^2 = a^2 + b^2 - 2ab \cos C, \qquad \text{Art. 82 (1)}$$

which may be adapted to logarithmic calculation by using the result of Ex. 4, Art. 154. Thus,

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cos C \\ &= (a+b)^2 \cos^2 \theta, \text{ if } \sin \theta = \frac{2\sqrt{ab}}{a+b} \cos \frac{C}{2}, \end{aligned}$$

or $$c = (a+b) \cos \theta.$$

We shall now employ this method to solve the example in Art. 191.

We have $$\sin\theta = \frac{2\sqrt{ab}}{a+b}\cos\frac{C}{2}.$$

Then $$L\sin\theta = \log 2 + \tfrac{1}{2}(\log a + \log b) - \log(a+b) + L\cos\frac{C}{2}$$
$$= \log 2 + \tfrac{1}{2}(\log 723 + \log 259) - \log 982 + L\cos 17^\circ\, 39'$$
$$= 9.92420$$

or $$\theta = 57^\circ\, 7'\, 26''.$$

Then $$\log c = \log(a+b) + L\cos\theta - 10$$
$$= \log 982 + L\cos 57^\circ\, 7'\, 26'' - 10$$
$$= 2.72678$$

and $$c = 533.06 \text{ as before.}$$

194. *Case IV.*—Given **two sides** and the **angle opposite** one of them, a, b, A. Euc. VI., 7.

From the Sine Rule, $\sin B = \dfrac{b\sin A}{a}$, Art. 80.

therefore $$L\sin B = \log b + L\sin A - \log a,$$

from this equation B may be found.

Then $$C = 180^\circ - (A+B), \text{ which gives } C,$$

then $$c = \frac{b\sin C}{\sin B},$$

or $$\log c = \log b + L\sin C - L\sin B,$$

from which c may be found.

Ex.—Given $a = 379.5$, $b = 564.8$, $A = 40^\circ\, 32'\, 16''$; find B, C, c.

$$L\sin B = \log 564.8 + L\sin 40^\circ\, 32'\, 16'' - \log 379.5$$
$$= 2.75189 + 9.81288 - 2.57921$$
$$= 9.98556,$$

from which $B = 75^\circ\, 18'\, 29''$, or $104^\circ\, 41'\, 31''$.

Then $\quad C = 180^\circ - (A + B) = 64^\circ\ 9'\ 15''$, or $34^\circ\ 46'\ 13''$,

also $\quad \log c = \log 564.8 + L \sin 64^\circ\ 9'\ 15'' - L \sin 75^\circ\ 18'\ 29''$

$$= 2.75189 + 9.95422 - 9.98556$$

$$= 2.72055,$$

therefore $c = 525.48$.

Taking the other value of C, we get

$$\log c = \log 564.8 + L \sin 34^\circ\ 46'\ 13'' - L \sin 75^\circ\ 18'\ 29''$$

$$= 2.75189 + 9.75609 - 9.98556$$

$$= 2.52242$$

from which $\quad c = 332.98$.

195. The preceding article contains what is known as the ambiguous case in the solution of triangles. Its peculiarities are fully discussed in Ex. 3, Art. 87, and in Art. 88, which the student should carefully examine. From Fig. 3, page 84, it will be observed that the difference between the two values of c is equal to BB', that is to $2a \cos B$ when B has the acute value. This gives a test of the accuracy of the work. Thus

$$a \cos B = 379.5 \times .25362 = 96.2498.$$

Half the difference of the values of $\quad c = 96.25$.

This proves the work sufficiently accurate.

196. In the solution of problems in which the trigonometrical tables are employed we should be careful to select the method which is most appropriate. An examination of the tables will show that for the natural functions when the angle is small, the sine and the tangent are the most appropriate. For the differences in the cosines are too small and those of the cotangent are irregular, so that in both cases the principle of proportional parts fails. The reverse is, of course, true, when the angle is nearly equal to a right angle. For the logarithmic functions the principle of proportional parts fails for both extremities of

the quadrant, though it holds good throughout the central part. We, therefore, seek for methods of solution not involving very small angles nor those nearly equal to a right angle.

For example, when the three sides of a triangle are given the angles may be found from one of the half angle formulæ, or from the sine of the whole angle. If a particular angle is known to be small we should choose the latter method, but if it is very near a right angle we should choose the former. Again, suppose we desire to find an angle from its cosine which is small so that the angle is nearly equal to a right angle. Then since

$$\cos A = n$$

we have

$$\tan \frac{A}{2} = \sqrt{\frac{1 - \cos A}{1 + \cos A}} = \sqrt{\frac{1 - n}{1 + n}},$$

and this formula is free from objections. Such transformations, however, are rarely necessary. In practical work, measurements of length containing four significant figures, and angular measurement to the nearest minute, are considered very accurate. The results obtained from using the tables given in the ordinary way will more than satisfy these conditions.

EXERCISE XXV.

1. The sides of a triangle are 350, 400, 450; find all the angles.

2. The sides of a triangle are 1263, 1359, 1468; find all the angles.

3. The sides of a triangle are 52.317, 24.659, 47.932; find the greatest angle.

4. Given $A = 31° 13'$, B $48° 24' 15''$, $a = 926.7$; solve the triangle.

5. Given $b = 7235$, $c = 1592$, $A = 50°$; solve the triangle.

6. Given $B = 25° 37'$, $C = 15° 23'$, $c = 259$; solve the triangle.

7. Given $b = 354$, $c = 426$, $A = 49° 16'$; solve the triangle.

8. Given $a=156.5$, $c=53.94$, $B=15° \ 13' \ 14''$; solve the triangle.

9. Given $a=325$, $b=333$, $A=52° \ 19'$; solve the triangle.

10. Given $c=1249.6$, $a=397.3$, $A=8° \ 19' \ 35''$; solve the triangle.

11. Given $c=432$, $a=135.17$, $A=18° \ 14'$; solve the triangle.

12. Given $B=37° \ 20'$, $b=4570$, $c=7563$; solve the triangle.

13. Given $C=152° \ 54' \ 20''$ $c=1249.6$, $a=397.3$; solve the triangle.

14. Given $b=9268$, $c=6951$, $A=16° \ 15' \ 38''$; find B and C, from $\log 7=.845098$, $L \cot 8° \ 7' \ 49''=10.8450980$.

15. Given $a=197$, $b=250$, $c=448$; find the angles.

16. Given $A=125° \ 30'$, $b=750$, $c=250$; find a without finding B or C.

17. In Ex. 16. find the segments of a formed by drawing a perpendicular from A.

18. Given $a=123.5$, $b=167.38$, $c=250$; find the area of the triangle.

19. The sides of a triangle are in the ratio of $2:3$, and the contained angle is $60°$; find the remaining angles. If the area is 100 square feet, find the remaining side.

20. In a triangle AD is drawn perpendicular to the base, and $BD=25$, $DC=40$, $AD=75$; find the angles.

21. The base BC of a triangle is 37.54, the perpendicular from the vertex on the base is 100, $L \sin B=9.68357$; solve the triangle.

22. In the ambiguous case A, b and a have fixed values, the latter being 88.34; the difference of the two values of C is $26° \ 30'$; find the difference between the areas of the two triangles.

23. Two angles of a triangle are $71° \ 28' \ 6''$, and $50° \ 56' \ 10''$, the greatest side is 2264; find the least side.

24. The sides of a triangle are 25.3, 40.7 and 50; find the radii of the inscribed, escribed and circumscribed circles.

25. Two angles of a triangle are 37° 20′, and 68° 40′, and the radius of the inscribed circle is 100; find the sides of the triangle.

26. In the preceding example find the sides if the radius of the circumscribing circle is 100.

27. The radii of the inscribed, circumscribed, and one escribed circle of a triangle are 2.8284, 5.8336, and 7.071 respectively; find the sides of the triangle.

28. Given the sides of a triangle, show that the angles may be found from the equations

$$\cos\frac{A-B}{2}=\frac{(a+b)\sin\theta}{2\sqrt{ab}},\ \sin\frac{C}{2}=\frac{c\sin\theta}{2\sqrt{ab}},$$

where $a-b=c\cos\theta$.

29. Given the sides of a triangle, show that the angles may be found from the equations

$$x+y=a,\quad \log(x-y)=\log(c+b)+\log(c-b)-\log a,$$
$$\log\cos B=\log x-\log c,\quad \log\cos C=\log y-\log b.$$

CHAPTER XIII.

PROPERTIES OF PLANE FIGURES.

197. In the preceding chapters simple examples have been given of the various parts of the subject usually treated in works on elementary trigonometry. We now propose to extend the principles already given to problems of a somewhat more difficult character.

Sides and Angles of Triangles.

198. Many important identities have been established involving the sides and angles of a triangle. Examples are given of the more useful transformations.

Ex. 1.—Prove $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$.

$$\begin{aligned} \sin A + \sin B + \sin C &= 2 \sin \frac{A}{2} \cos \frac{A}{2} + 2 \sin \frac{B+C}{2} \cos \frac{B-C}{2} \\ &= 2 \cos \frac{A}{2}\left(\cos \frac{B+C}{2} + \cos \frac{B-C}{2}\right) \\ &= 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}. \end{aligned}$$

In the above, $\sin \frac{B+C}{2}$ is replaced by $\cos \frac{A}{2}$, and $\sin \frac{A}{2}$ by $\cos \frac{B+C}{2}$, since $\frac{B+C}{2}$ and $\frac{A}{2}$ are complementary angles. Such interchanges are very frequently employed in transforming expressions involving the angles of a triangle.

Ex. 2.—Prove $\tan A+\tan B+\tan C=\tan A\tan B\tan C$.

$$\tan A+\tan B+\tan C=\frac{\sin A}{\cos A}+\frac{\sin(B+C)}{\cos B\cos C}$$

$$=\sin A\left\{\frac{1}{\cos A}+\frac{1}{\cos B\cos C}\right\}$$

$$=\sin A\left\{\frac{\cos B\cos C-\cos(B+C)}{\cos A\cos B\cos C}\right\}$$

$$=\frac{\sin A\sin B\sin C}{\cos A\cos B\cos C}=\tan A\tan B\tan C$$

In the above $\sin(B+C)$ is replaced by $\sin A$, and $\cos A$ by $-\cos(B+C)$, these being supplementary angles.

Ex. 3.—Prove

$$(b+c-a)\tan\frac{A}{2}=(c+a-b)\tan\frac{B}{2}=(a+b-c)\tan\frac{C}{2}.$$

With the usual notation we have

$$(b+c-a)\tan\frac{A}{2}=2(s-a)\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=2\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

The symmetry of this result shows that it will remain unchanged after any symmetrical interchange of the sides and angle.

199. In the solution of many problems it is convenient to express the sides of a triangle in terms of the opposite angles.

Ex. 1.—Prove $(b-c)\cot\frac{A}{2}+(c-a)\cot\frac{B}{2}+(a-b)\cot\frac{C}{2}=0$.

Let $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$$

then $$a=k\sin A,\ b=k\sin B,\ c=k\sin C.$$

And $(b-c)\cot\frac{A}{2}+(c-a)\cot\frac{B}{2}+(a-b)\cot\frac{C}{2}$

$$=k\left\{(\sin B-\sin C)\cot\frac{A}{2}+(\sin C-\sin A)\cot\frac{B}{2}\right.$$
$$\left.+(\sin A-\sin B)\cot\frac{C}{2}\right\}$$
$$=2k\left\{\sin\frac{B-C}{2}\sin\frac{B+C}{2}+\sin\frac{C-A}{2}\sin\frac{C+A}{2}\right.$$
$$\left.+\sin\frac{A-B}{2}\sin\frac{A+B}{2}\right\}$$
$$=2k\left\{\sin^2\frac{B}{2}-\sin^2\frac{C}{2}+\sin^2\frac{C}{2}-\sin^2\frac{A}{2}+\sin^2\frac{A}{2}-\sin^2\frac{B}{2}\right\}$$
$$=0.$$

Ex. 2.—If a^2, b^2, c^2 are in A. P., then $\cot A$, $\cot B$, $\cot C$ are also in A. P.

Since a^2, b^2, c^2 are in **A.** P. we have

$$b^2=a^2-b^2+c^2$$

or $$k^2\sin^2 B=k^2(\sin^2 A-\sin^2 B+\sin^2 C)$$

or $$\sin^2 B=\sin(A+B)\sin(A-B)+\sin(A+B)\sin C$$
$$=\sin C\{\sin(A-B)+\sin(A+B)\}$$
$$=2\sin C\sin A\cos B.$$

Dividing by $\sin A\sin B\sin C$ and interchanging sides, we get

$$2\cot B=\frac{\sin B}{\sin C\sin A}=\frac{\sin(C+A)}{\sin C\sin A}=\cot A+\cot C,$$

which proves the proposition.

200. If the terms of a ratio or the two sides of an equation are homogeneous functions of the same number of dimensions of the sides of a triangle, the sides may be replaced by the sines of opposite angles. The truth of this statement will be evident from the examples of the preceding article. A formal proof, however, may easily be given from Art. 34, of the High School Algebra, Part II.

We give two further examples illustrating important principles.

Ex. 1.—Find the greatest value of $\cos A \cos B \cos C$ in which A, B, C are the angles of a triangle.

Suppose C to have any fixed value, then $A+B$ is constant, and we have

$$2 \cos A \cos B = \cos (A+B) + \cos (A-B)$$

for all values of A and B. But since $A+B$ is constant and $\cos (A-B)$ is greatest when $A-B$ is 0, therefore $\cos A \cos B$ is greatest when $A=B$. Similarly for any value of A, $\cos B \cos C$ is greatest when $B=C$, etc. Hence $\cos A \cos B \cos C$ is greatest when $A=B=C=60°$, and then $\cos A \cos B \cos C = \frac{1}{8}$.

Ex. 2.—Factor $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C - 1$, in which A, B, C are any angles whatever.

$$\begin{aligned}
&\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C - 1\\
&\quad = (\cos A + \cos B \cos C)^2 + \cos^2 B + \cos^2 C - 1 - \cos^2 B \cos^2 C\\
&\quad = (\cos A + \cos B \cos C)^2 - (1 - \cos^2 B)(1 - \cos^2 C)\\
&\quad = (\cos A + \cos B \cos C)^2 - \sin^2 B \sin^2 C\\
&\quad = (\cos A + \cos B \cos C + \sin B \sin C)(\cos A + \cos B \cos C\\
&\qquad\qquad - \sin B \sin C)\\
&\quad = \{\cos A + \cos (B-C)\}\{\cos A + \cos (B+C)\}\\
&\quad = 4 \cos \frac{A+B+C}{2} \cos \frac{A-B+C}{2} \cos \frac{A+B+C}{2} \cos \frac{B+C-A}{2}.
\end{aligned}$$

If A, B, C are the angles of a triangle this expression vanishes. It also vanishes if any one of the four compound angles is an odd multiple of a right angle.

201. Since three of the six elements of a triangle, one of them being a side, are sufficient to determine the remaining elements, it follows that not more than three independent relations can exist between the sides and angles. Also from any three independent relations all the others may be found. The following

are three different groups of such relations from any one of which the others may be derived:

$$\text{I.} \begin{cases} \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}, \\ A + B + C = \pi. \end{cases}$$ Art. 80.

$$\text{II.} \begin{cases} a = b \cos C + c \cos B, \\ b = c \cos A + a \cos C, \\ c = a \cos B + b \cos A. \end{cases}$$ Art. 81.

$$\text{III.} \begin{cases} a^2 = b^2 + c^2 - 2bc \cos A, \\ b^2 = c^2 + a^2 - 2ca \cos B, \\ c^2 = a^2 + b^2 - 2ab \cos C. \end{cases}$$ Art. 82.

1. To derive II. from I.

From $A + B + c = \pi,$

we have $\sin A = \sin (B + C)$

$= \sin B \cos C + \cos C \sin B,$

or, $a = b \cos C + c \cos B$

by substituting a, b, c, for $\sin A$, $\sin B$, $\sin C$. Art. 200.

2. To derive III. from I.

From $A + B + C = \pi$, $\sin A = \sin (B + C)$, $\cos A = -\cos (B + C)$.

Then $\sin^2 A = \sin^2 B \cos^2 C + 2 \sin B \cos C \cos B \sin C + \cos^2 B \sin^2 C$

$= \sin^2 B + \sin^2 C + 2 \sin B \sin C (\cos B \cos C - \sin B \sin C)$

$= \sin^2 B + \sin^2 C - 2 \sin B \sin C \cos A,$

or $a^2 = b^2 + c^2 - 2bc \cos A.$ Art. 200.

3. To derive I. from II.

From the three equations find $\cos A$, $\cos B$, $\cos C$, in terms of the sides; thence find $\sin A$, $\sin B$, $\sin C$, and the Sine Rule follows immediately. Again from the same equations eliminate a, b, c, and we obtain

$$\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C - 1 = 0,$$

and consequently one of its factors, as given in the preceding article, must vanish. From the consideration that each angle of a triangle is positive, and the sum of each pair is less than 180°, we find that it must be $\cos \frac{1}{2}(A+B+C)$ which vanishes, for the value $\frac{\pi}{2}$. Thus $A+B+C=\pi$. The student should make all the remaining transformations, none of which present any difficulty.

EXERCISE XXVI.

Prove the following identities in which A, B, C are the angles of a triangle:

1. $\sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}$.

2. $\sin A - \sin B + \sin C = 4 \sin \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$.

3. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$.

4. $\sin 2A - \sin 2B + \sin 2C = 4 \cos A \sin B \cos C$.

5. $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$.

6. $\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$.

7. $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$.

8. $\cos A + \cos B + \cos C - 1 = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$.

9. $\cos 2A + \cos 2B + \cos 2C + 4 \cos A \cos B \cos C + 1 = 0$.

10. $\sin(B+C-A) + \sin(C+A-B) + \sin(A+B-C)$
$= 4 \sin A \sin B \sin C$.

11. $\sin^2 A - \sin^2 B + \sin^2 C = 2 \sin A \cos B \sin C$.

12. $\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} - \cos^2 \frac{C}{2} = 2 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$.

13. $\sin^2 A = \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C$.

14. $\cos^2 A + \cos^2 B + 2 \cos A \cos B \cos C = \sin^2 A + \sin^2 B$
$- 2 \sin A \sin B \cos C$.

In a triangle right-angled at C, prove the following:

15. $\cos^2 \dfrac{A}{2} = \dfrac{b+c}{2c}$.

16. $\sec 2B = \dfrac{a^2+b^2}{a^2-b^2}$.

17. $\tan 2B = \dfrac{2 \sin A \sin B}{\sin^2 A - \sin^2 B}$.

18. $\operatorname{cosec} 2B = \dfrac{a}{2b} + \dfrac{b}{2a}$.

19. $a^3 \cos A + b^3 \cos B = abc$.

20. $abc^2 \sin A \sin B = 4S^2$

In any triangle prove the following:

21. $\cot \dfrac{A}{2} \cot \dfrac{B}{2} = \dfrac{a+b+c}{a+b-c}$.

22. $\dfrac{\tan \frac{1}{2} B}{\tan \frac{1}{2} A} = \dfrac{b+c-a}{c+a-b}$.

23. $\dfrac{\tan B}{\tan C} = \dfrac{a^2+b^2-c^2}{a^2-b^2+c^2}$.

24. $\dfrac{\sin (A-B)}{\sin C} = \dfrac{a^2-b^2}{c^2}$.

25. $\dfrac{a-b}{c} = \dfrac{\cos B - \cos A}{1+\cos C}$.

26. $\dfrac{b+c}{a} = \dfrac{\cos B + \cos C}{1 - \cos A}$.

27. $a \sin (B-C) + b \sin (C-A) + c \sin (A-B) = 0$.

28. $2a (\sin C - \cos B \sin A) = b \sin 2A$.

29. $a (b \cos C - c \cos B) = b^2 - c^2$.

30. $(a^2-b^2) \cot C + (b^2-c^2) \cot A + (c^2-a^2) \cot B = 0$.

31. $\sin A (\cos A + \cos B \cos C) = \sin B (\cos B + \cos C \cos A)$.

32. $b (\tan B + \tan C) = a \tan B \sec C$.

33. $(a+b) \cos C + (b+c) \cos A + (c+a) \cos B = a+b+c$.

34. $a \cos A + b \cos B + c \cos C = 2a \sin B \sin C = 2b \sin C \sin A$.

35. $2ab \cos C + 2bc \cos A + 2ca \cos B = a^2+b^2+c^2$.

36. $a \sec A - b \sec B = \sec C (b \sec A - a \sec B)$.

37. $a^2 - 2ab \cos (60° + C) = c^2 - 2bc \cos (60° + A)$.

38. The perimeter of any triangle is $2c \cos \dfrac{A}{2} \cos \dfrac{B}{2} \operatorname{cosec} \dfrac{C}{2}$.

39. The greatest value of $\sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}$ is $\dfrac{1}{8}$.

40. If $y \sin^2 A + x \sin^2 B = z \sin^2 B + y \sin^2 C = x \sin^2 C + z \sin^2 A$

then $$\frac{x}{\sin 2A} = \frac{y}{\sin 2B} = \frac{z}{\sin 2C}.$$

41. $$\frac{\tan A}{\tan B}+\frac{\tan B}{\tan C}+\frac{\tan C}{\tan A}+\frac{\tan A}{\tan C}+\frac{\tan B}{\tan A}+\frac{\tan C}{\tan B}+2$$
$$=\sec A\sec B\sec C.$$

42. If a, b, c are in A. P., then $\cos A$, $\operatorname{vers} B$, $\cos C$ are also in A. P.

43. If a^2+bc, b^2+ca, c^2+ab are in A. P., then $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ are also in A. P.

44. Factor $1-\cos^2 A-\cos^2 B-\cos^2 C+2\cos A\cos B\cos C$ in which A, B, C are any angles whatever.

Triangles and Circles.

202. A triangle may be solved from various data. In general any three independent measurements, one of which is a length, are sufficient. We give two examples.

Ex. 1.—Given two sides of a triangle and the line bisecting the included angle to solve the triangle.

Let b, c be the given sides; l the bisector of the angle A which meets the base in D.

We have $$BD:DC=c:b \qquad \text{Euc. VI., 3.}$$

from which $$BD=\frac{ac}{b+c}.$$

Then in triangle ABD we have from the Sine Rule:

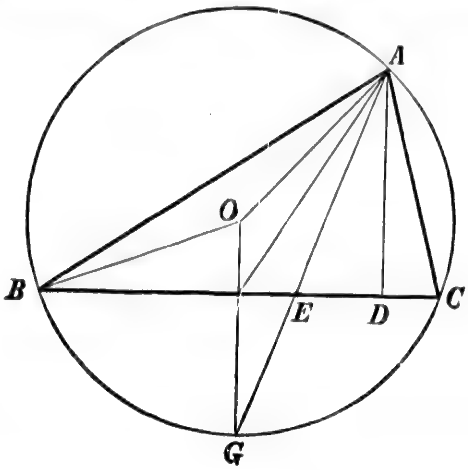
$$\frac{l}{\sin B}=\frac{BD}{\sin\frac{1}{2}A}=\frac{ac}{(b+c)\sin\frac{1}{2}A},$$

or $$l=\frac{ac\sin B}{(b+c)\sin\frac{1}{2}A}=\frac{bc\sin A}{(b+c)\sin\frac{1}{2}A}=\frac{2bc\cos\frac{A}{2}}{b+c}.$$

Therefore $$\cos\frac{A}{2}=\frac{(b+c)\,l}{2bc}.$$

Two sides and the included angle are now known, which is Case III. of the preceding chapter.

Ex. 2.—Given the perpendicular from the vertex of a triangle on the base, the bisector of the vertical angle and the line joining the vertex to the centre of the base, to solve the triangle.



In triangle ABC, let $AD = d$, $AE = e$, $AF = f$ and $OA = OB = R$, the radius of the circumscribing circle.

Then $$\angle OAG = \angle OGA = \angle GAD = \cos^{-1}\frac{d}{e}$$

and $$\begin{aligned} \angle OAF &= \angle OAE - \angle FAE \\ &= \angle EAD - (\angle FAD - \angle EAD) \\ &= 2\angle EAD - \angle FAD \\ &= 2\cos^{-1}\frac{d}{e} - \cos^{-1}\frac{d}{f}. \end{aligned}$$

From the triangle OAF we have,

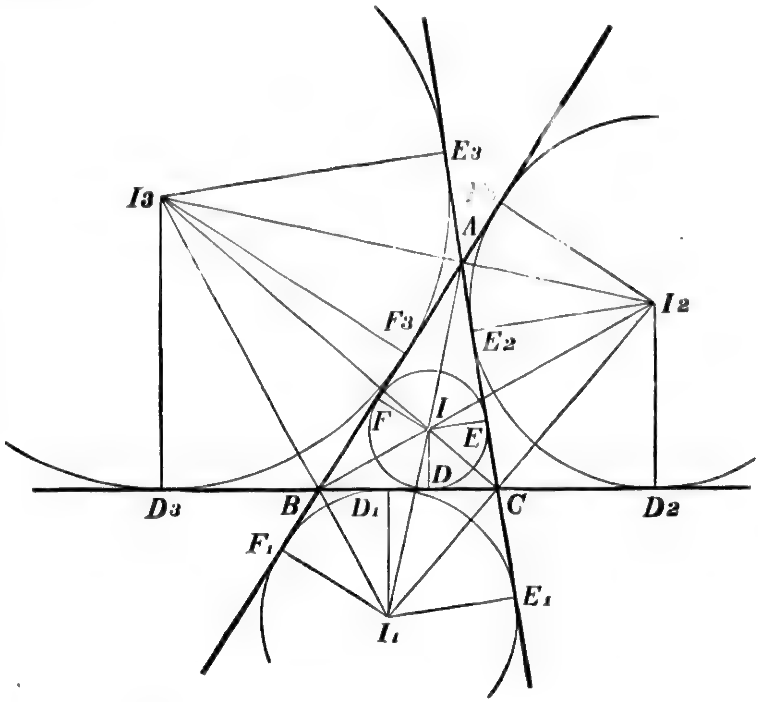
$$\frac{OF}{\sin OAF} = \frac{OA}{\sin OFA} = \frac{OB}{\sin FAD},$$

from which $$\frac{OF}{OB} = \frac{\sin OAF}{\sin FAD},$$

that is $$\cos A = \frac{\sin\left(2\cos^{-1}\frac{d}{e} - \cos^{-1}\frac{d}{f}\right)}{\sin\left(\cos^{-1}\frac{d}{f}\right)}.$$

From this equation A is determined and then $B+C$ is known. Also the angle EAD may easily be shown to be equal to $\frac{1}{2}(C-B)$, and thus B and C may be found. The remainder of the solution presents no difficulty.

203. The following are a few of the many interesting properties connected with the inscribed and escribed circles of a triangle. We assume the results already proved in Arts. 91 and 93.



1. The centre of a circle lies on the line bisecting the angle between two intersecting tangents. The following sets of points are therefore collinear:

$$A, I, I_1; \quad B, I, I_2; \quad C, I, I_3; \quad I_2, A, I_3; \quad I_3, B, I_1; \quad I_1, C, I_2.$$

2. Tangents drawn from the same point to a circle are equal. The truth of the following may therefore be easily shown.

(1) $AF = AE = s - a,\ AF_1 = AE_1 = s.$ Arts. 91, 93.

(2) $AF_3 = AE_3 = CE_3 - CA = s - b.$

(3) $AF_2 = AE_2 = BF_2 - BA = s - c.$

(4) $FF_1 = AF_1 - AF = s - (s - a) = a.$

(5) $F_1F_2 = AF_1 + BF_2 - AB = 2s - c = a + b.$

(6) $AF + AF_1 + AF_2 + AF_3 = a + b + c.$

(7) $F_3F = AF - AF_3 = (s - a) - (s - b) = b - a.$

(8) $II_1 = AI_1 - AI = (AF_1 - AF) \sec \frac{A}{2} = a \sec \frac{A}{2}.$

(9) $I_1C = E_1C \sec E_1CI_1 = (s - b) \operatorname{cosec} \frac{C}{2}.$

(10) $I_1I_2 = I_1C + CI_2 = (s - b) \operatorname{cosec} \frac{C}{2} + (s - a) \operatorname{cosec} \frac{C}{2} = c \operatorname{cosec} \frac{C}{2}.$

204. *To express the radii of the inscribed and escribed circles in terms of the radius of the circumscribed circle.*

From the triangle BIC we have,

$$\frac{IC}{\sin \frac{1}{2} B} = \frac{BC}{\sin BIC} = \frac{a}{\sin \frac{1}{2}(B + C)} = \frac{a}{\cos \frac{1}{2} A},$$

then
$$r = IC \sin \frac{C}{2} = \frac{a \sin \frac{1}{2} B \sin \frac{1}{2} C}{\cos \frac{1}{2} A}$$

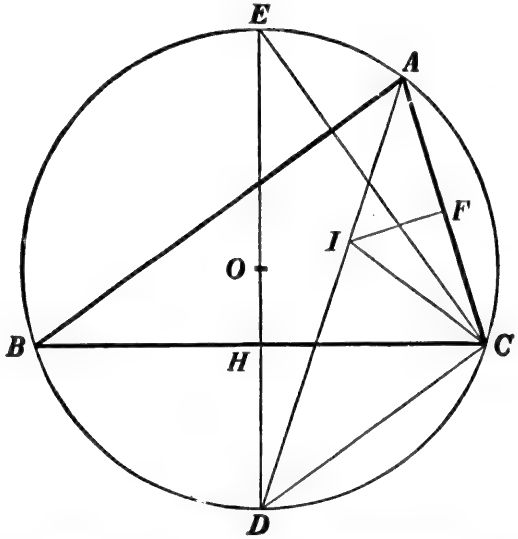
$$= 4 R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}. \qquad \text{Art. 90 (1).}$$

Similarly from the triangles BI_1C, CI_2A, AI_3B, we obtain,

$$r_1 = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2},\quad r_2 = 4R \sin \frac{B}{2} \cos \frac{C}{2} \cos \frac{A}{2},$$

$$r^3 = 4R \sin \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2}.$$

205. *The rectangle of the segments of any chord of the circumscribed circle of a triangle drawn through the centre of the inscribed circle is equal to twice the rectangle contained by their radii.*



Draw the diameter EOD of the circumscribing circle perpendicular to BC; join DA, CE. Then DA bisects the angle A and consequently contains the centre, I, of the inscribed circle. Draw IF perpendicular to AC, then $IF=r$, the radius of the inscribed circle.

Now $\qquad \angle DIC = \angle IAC + \angle ICA = \frac{1}{2}(A+C)$

and $\qquad \angle DCI = \angle DCB + \angle BCI = \frac{1}{2}(A+C)$

therefore $\qquad DC = DI.$

The triangles DEC, IAF, are similar, since each contains a right angle, and the angles at E and A stand on the same arc.

Therefore $\qquad DE : DC = IA : IF, \qquad$ Euc. VI., 4.

therefore $\qquad DE \,.\, IF = DC \,.\, IA \qquad$ Euc. VI., 16.

that is $\qquad 2Rr = DI \,.\, IA.$

But the rectangle $DI.IA$ is equal to the rectangle of the segments of any chord drawn through I, which completes the proposition. Euc. III., 35.

206. *To find the distance between the centres of the inscribed and circumscribed circles of a triangle.*

Draw a diameter through O and I, meeting the circle in H and K.

Then $HI.IK = DI.IA$ Euc. III., 35.

or $(R+OI)(R-OI) = 2rR$ Art. 205.

from which $OI^2 = R^2 - 2rR.$

Cor. Since OI^2 is essentially positive R can never be less than $2r$.

207. *The circumscribed circle of a triangle bisects the lines joining the centres of the inscribed circle with each of the escribed circles.*

For $DI = DC = HC \sec BCD = \frac{a}{2} \sec \frac{A}{2}.$

But from Art. 203 (8), $II_1 = a \sec \frac{A}{2}$, which proves the proposition.

208. The point of intersection of the perpendiculars from the vertices of a triangle on the opposite sides is called the **ortho-centre**. The triangle formed by joining the feet of the perpendiculars is called the **pedal triangle.**

209. The diagram of Art. 203 proves indirectly a number of important properties of a triangle. We make the following observations:

1. In the triangle $I_1I_2I_3$, the lines I_1A, I_2B, I_3C, are perpendiculars from the vertices on the opposite sides and consequently ABC is the pedal triangle.

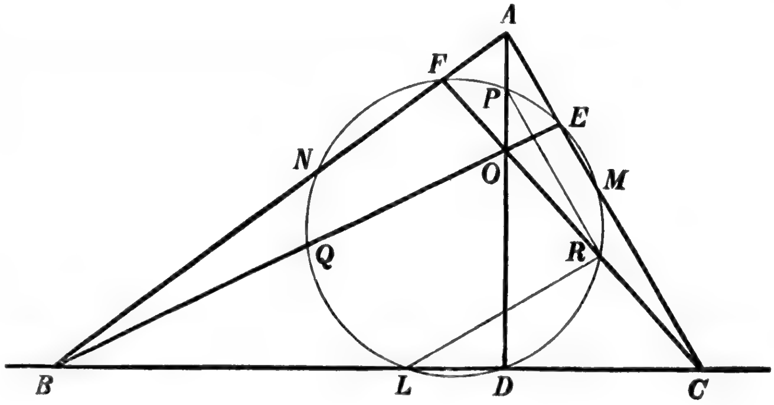
2. The orthocentre, I, of the triangle $I_1I_2I_3$ is the centre of the inscribed circle of the pedal triangle ABC.

3. The vertices of the triangle $I_1I_2I_3$ are centres of the escribed circles of the pedal triangle.

4. The circle circumscribing the pedal triangle ABC, bisects the lines joining the orthocentre and the vertices of the original triangle. (Art. 207).

We shall employ these principles to prove the interesting theorem of the next article.

210. *The middle points of the sides of a triangle, the feet of the perpendiculars from the opposite angles, and the points midway between the orthocentre and the vertices lie on a circle.*



Let the perpendiculars AD, BE, CF, meet in O; then DEF is the pedal triangle of ABC, and the circle passing through D, E, F, passes also through P, Q, R, the middle points of OA, OB, OC. (Arts. 207 and 209.)

Let L, M, N be the middle points of BC, CA, AB, and join PR, RL.

Since P, R, bisect OA, OC; PR is parallel to AC.
Similarly RL is parallel to BE;
therefore $\angle PRL = \angle BEC =$ a right angle.

The circle on PL as diameter passes through D and R. (Euc. III., 31.) But the six points D, E, F; P, Q, R, have been shown to lie on a circle; therefore the circle passing through three of them, P, R, D, passes through them all. This circle has been shown to pass through L, the centre of one of the sides; similarly it may be shown to pass through the centres of the remaining sides. Hence the nine points D, E, F; P, Q, R; L, M, N, lie on a circle.

EXERCISE XXVII.

1. In a triangle AD and AD' are the internal and external bisectors of the vertical angle; show that

$$AD = \frac{2bc \cos \frac{1}{2} A}{b+c}, \quad AD' = \frac{2bc \sin \frac{1}{2} A}{b-c}, \quad DD' = \frac{2abc}{b^2 - c^2}.$$

2. Given the vertical angle of a triangle, the perpendicular from the vertex on the base and the bisector of the vertical angle; show how to solve the triangle.

3. If l, m, n, are the bisectors of the angles of a triangle, then

$$l\,(2s-a) \sin \frac{A}{2} = m\,(2s-b) \sin \frac{B}{2} = n\,(2s-c) \sin \frac{C}{2}.$$

4. In the previous example show that

$$\frac{\cos \frac{1}{2} A}{l} + \frac{\cos \frac{1}{2} B}{m} + \frac{\cos \frac{1}{2} C}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

5. The perpendiculars from the angles of a triangle on the opposite sides are 4, 5, 6, find the greatest angle and the area.

6. AD is the median and h the altitude of the triangle ABC; express BD and DC in terms of h and the base angles, and thus prove $2 \cot D = \cot B - \cot C$.

7. Extend the results of the preceding example to show that $\cot DAB = 2 \cot A + \cot B$, $\quad \cot DAC = 2 \cot A + \cot C$.

8. In any triangle prove $a^2 = (b+c)^2 - 4bc \sin^2 \dfrac{A}{2}$, and thence show how to solve a triangle having given the base, the sum of the sides, and the altitude.

9. Solve a triangle, given the base, difference of the sides and the altitude.

10. Given the base, the vertical angle and the sum of the sides; solve the triangle. (From the Sine Rule $\cos \frac{1}{2}(B-C)$ may be obtained.)

11. In the figure of Art. 203, show that the circle described on II_1 as diameter passes through B and C, and that similarly described on I_1I_2 passes through A and B.

12. In the same figure prove $AE_3 = CE_1 = BD = s - b$. Find three other tangents of the same length.

13. Express the lengths of the various tangents from the vertices B and C in terms of the sides.

14. In the figure of Art. 210 show that $AO = 2R \cos A$, $OD = 2R \cos B \cos C$. Examine the case in which one of the angles involved is obtuse.

15. The medians of the triangle ABC meet in G; show that the distance of G from the side BC is $\frac{2}{3} R \sin B \sin C$, and that $AG = \frac{1}{3}\sqrt{2b^2 + 2c^2 - a^2}$.

16. Prove that the length of the perpendiculars from the point of intersection of the medians of a triangle on the side are inversely proportional to the sides.

17. Show that the distances of the orthocentre from the sides of a triangle are inversely proportioned to those of the centre of the circumscribed circle.

18. Show that the area of a triangle is given by each of the following expressions:

(1) $s(s-a) \tan \dfrac{A}{2}$. (2) $s^2 \tan \dfrac{A}{2} \tan \dfrac{B}{2} \tan \dfrac{C}{2}$.

(3) $\dfrac{a^2 \sin B \sin C}{2 \sin (B+C)}$.

(4) $\dfrac{2abc}{a+b+c} \cos \dfrac{A}{2} \cos \dfrac{B}{2} \cos \dfrac{C}{2}$.

(5) $\dfrac{2s^2 . \sin A \sin B \sin C}{\sin A + \sin B + \sin C}$.

(6) $\dfrac{(a+b+c)^2}{4(\cot \frac{1}{2} A + \cot \frac{1}{2} B + \cot \frac{1}{2} C)}$.

(7) $\dfrac{a^2+b^2+c^2}{4 (\cot A + \cot B + \cot C)}$.

(8) $2R^2 \sin A \sin B \sin C$.

(9) $\frac{1}{2} R(a \cos A + b \cos B + c \cos C) = \frac{1}{2} R^2(\sin 2A + \sin 2B + \sin 2C)$.

19. In any triangle prove the following relations:

(1) $\dfrac{rr_1}{r_2 r_3} = \tan^2 \dfrac{A}{2}$.

(2) $R \tan A = \dfrac{abc}{b^2+c^2-a^2}$.

(3) $r_1 + r_2 + r_3 = R\,(3 + \cos A + \cos B + \cos C)$.

(4) $R + r = R\,(\cos A + \cos B + \cos C)$.

(5) $\tan \dfrac{A}{2} + \tan \dfrac{B}{2} + \tan \dfrac{C}{2} + \dfrac{s}{r} = 4R\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right)$.

20 If the radius of the inscribed circle of a triangle be equal to half that of the circumscribed circle the triangle is equilateral.

21. The area of any triangle is $Rr\,(\sin A + \sin B + \sin C)$.

22. Show that the angles of the triangle $I_1I_2I_3$ are $\frac{1}{2}\,(B+C)$, $\frac{1}{2}\,(C+A)$, $\frac{1}{2}\,(A+B)$, and the sides $a \operatorname{cosec} \dfrac{A}{2}$, $b \operatorname{cosec} \dfrac{B}{2}$, $c \operatorname{cosec} \dfrac{C}{2}$.

23. Show that the radius of the nine-point circle of a triangle is half the radius of the circumscribing circle.

24. The angles of the pedal triangle DEF are $\pi - 2A$, $\pi - 2B$, $\pi - 2C$, and the sides are $a \cos A$, $b \cos B$, $c \cos C$.

25. The sum of the sides of the pedal triangle is $4R \sin A \sin B \sin C$, and the area is $\frac{1}{2}\, bc \cos B \cos C \sin 2A$.

26. The radius of the circle incribed in the pedal triangle is $2R \cos A \cos B \cos C$, and the radius of an escribed circle is $2R \cos A \sin B \sin C$.

27. Show that $r_1 + r_2 + r_3 - r = 4R$, and thence that the area of the triangle $I_1I_2I_3$ is $2Rs$.

28. Find the angles of a triangle whose sides are proportional to $\cos\frac{1}{2}A$, $\cos\frac{1}{2}B$, $\cos\frac{1}{2}C$.

29. In any triangle the sum of the reciprocals of the perpendiculars from the vertices on the opposite sides is equal to the reciprocal of the radius of the inscribed circle.

30. If the sides of a triangle are in A. P., the radii of the escribed circles are in H. P.

31. If the sides of a triangle are in H. P., the areas of its escribed circles are also in H. P.

32. The rectangle of the segments of any chord of the circumscribed circle of a triangle drawn through the centre of an escribed circle is equal to twice the rectangle of their radii.

33. Show that the distance between the centres of the circles in the preceding example is given by the equation $OI_1^2 = R^2 + 2r_1R$.

34. If the points of contact of the inscribed circle of a triangle be joined the sides of the triangle then formed will be $2r\cos\frac{1}{2}A$, $2r\cos\frac{1}{2}B$, $2r\cos\frac{1}{2}C$, and its area $2r^2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$.

35. The sides of a triangle are $a+b$, $b+c$, $c+a$, show that the square of the radius of the inscribed circle is $2rR$.

36. In the ambiguous case in the solution of a triangle when a, b, A are given, show that the circles circumscribing the two triangles are equal, and the distance between their centres is $\sqrt{a^2\operatorname{cosec}^2 A - b^2}$.

37. If an equilateral triangle have its angular points in three parallel straight lines, of which the middle one is distant from the outside ones by a and b, then its side is $\frac{2}{3}\sqrt{3(a^2+ab+b^2)}$.

38. If the sides of a triangle are in A. P., show that its area is $\frac{1}{b}\sqrt{b^2-4(a-b)^2}$ times the area of an equilateral triangle of the same perimeter.

39. If p, q, r are the perpendiculars from the vertices of a triangle on the opposite sides, then

$$\frac{p^2}{qr}+\frac{q^2}{rp}+\frac{r^2}{pq}=\frac{bc}{a^2}+\frac{ca}{b^2}+\frac{ab}{c^2}.$$

CHAPTER XIV.

THE MEASUREMENT OF HEIGHTS AND DISTANCES.

211. An important application of trigonometry is the computation of the distance between two points without actual measurement. The direct measurement of the straight line joining two remote points is always a tedious and difficult operation, and very frequently impossible. But the measurement of the angle between the lines drawn from the observer's eye to any two visible points can, with proper instruments, be made with facility and precision. For this reason it is usual to make calculations depend upon the observation of angles so far as possible, but the measurement of one length will always be necessary. In order, then, to compute the distance between two points we require to measure directly

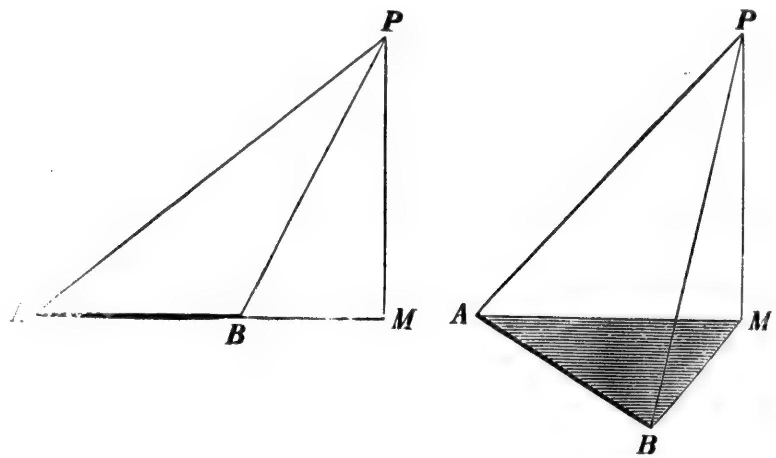
1. The distance between a pair of *accessible* points.

2. The angle between the lines drawn from the eye to some pair of *visible* points.

212. *To find the vertical height of an accessible object on a horizontal plane.*

From the base of the object measure a horizontal line of any length, d, and from its extremity observe the angle of elevation, A, of the top of the object. The height, h, is then given by the equation $h = d \tan A$. This and the equivalent equation, $d = h \cot A$, are very frequently employed.

213. *To find the height and distance of an inaccessible object on a horizontal plane.*



Let $PM = h$ be the vertical height; A and B two convenient points whose distance, d, from each other can be measured, and from which the angles of elevation of P can be observed.

1. If A, B, M are in a straight line, we have,

$$h \cot A = MA,\ h \cot B = MB,$$

from which $$h(\cot B - \cot A) = MA - MB = d,$$

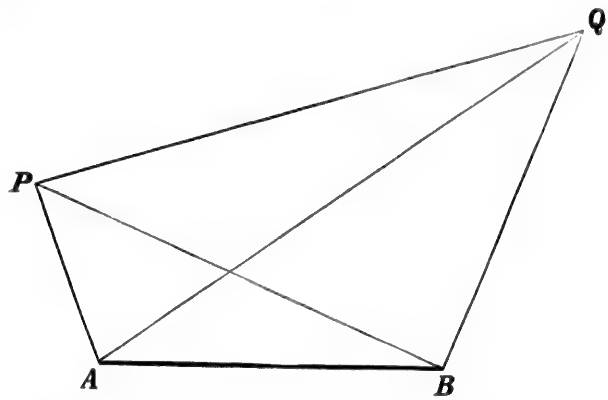
or, $$h = \frac{d}{\cot B - \cot A}$$

and then $$MA = h \cot A = \frac{d \cot A}{\cot B - \cot A}.$$

2. If A, B, M are not in a straight line, the angles PAB, PBA must also be observed. Then in the triangle PAB two angles and a side are known from which AP can be calculated. Lastly, in the right-angled triangle PAM, the side AP and the angle PAM are known, from which AM and MP can be found.

In the second case the line AB is not necessarily horizontal.

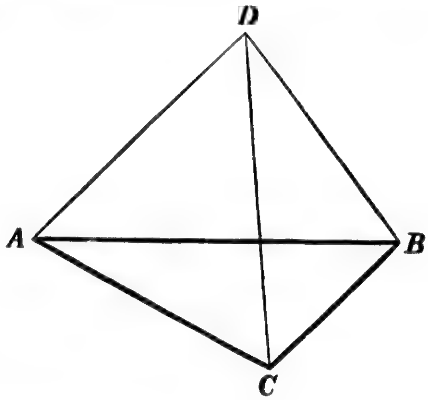
214. *To find the distance between two visible, but inaccessible objects.*



Let P and Q be the two objects, A and B two convenient points of observation. Measure the distance AB. At A and B observe the angles PAB, QAB; PBA, QBA. And if the triangles PAB, QAB are not in the same plane, observe the angle PAQ. Then in the triangle PAB; AB, $\angle PAB$, $\angle PBA$ are known, and therefore AP can be calculated. And in triangle QAB; AB, $\angle QAB$, $\angle QBA$ are known, and therefore AQ can be calculated. Then in triangle PAQ; AP, AQ, $\angle PAQ$ are known, and therefore PQ can be calculated.

215. In working problems which involve points above the horizontal plane it is often useful to employ the figure obtained by projecting such points, together with the lines connected with them, upon the plane. If, for example, A and B are any two points of observation in the horizontal plane, P, a point above it, we draw the perpendicular PN, join AN, BN; then instead of the triangle APB, of which one point is above the plane, we have ANB lying on the plane. Also if h be the height of PN and α, β the angles of elevation at A, B, we have $AN = h \cot \alpha$, $BN = h \cot \beta$. The diagram is thus much simplified by bringing all lines and points involved into a single plane.

216. A, B, C *are the angular points of a triangle, the lengths of whose sides are known;* D *is a point in the plane of the triangle at which the sides* AC, BC *subtend given angles* α, β. *It is required to find the distances* AD, BD.



Let the angles CAD, CBD be denoted by x, y respectively; then, since the sum of the angles of the quadrilateral $ACDB$ is four right angles, and the sum of the angles ADB, ACB is $\alpha+\beta+C$, we have

$$x+y=2\pi-\alpha-\beta-C$$

so that the sum of x and y is known. Then from the triangles ADC, BDC we have

$$CD=\frac{b\sin x}{\sin\alpha}=\frac{a\sin y}{\sin\beta};$$

from which
$$\frac{\sin x}{\sin y}=\frac{a\sin\alpha}{b\sin\beta}.$$

Now take a subsidiary angle ϕ, such that

$$\tan\phi=\frac{a\sin\alpha}{b\sin\beta}=\frac{\sin x}{\sin y},$$

and we obtain

$$\frac{\sin x-\sin y}{\sin x+\sin y}=\frac{\tan\phi-1}{\tan\phi+1}=\tan(\phi-45^\circ)$$

from which

$$\tan\frac{x-y}{2}=\tan\frac{x+y}{2}\tan(\phi-45^\circ).$$

This equation gives the value of $x-y$, and since $x+y$ is known, x and y can be found. The remainder of the solution presents no difficulty.

EXERCISE XXVIII.

1. At distances of a and b feet from the foot of a tower on a horizontal plane the angles of elevation are complementary; find the height.

2. The angle of elevation of a tower at a distance of 400 feet is half its elevation at 150 feet; find its height.

3. A tower leans towards the north with an inclination θ, and at a distance a to the south on a horizontal plane, the elevation of its summit is α; show that its vertical height and its slant height are $\dfrac{a}{\cot\alpha-\cot\theta}$, and $\dfrac{a\operatorname{cosec}\theta}{\cot\alpha-\cot\theta}$.

4. A spherical balloon whose diameter is d subtends an angle α at the eye of a spectator, whilst the angular elevation of its centre is β; show that the height of its centre is $\frac{1}{2}d\sin\beta\operatorname{cosec}\dfrac{\alpha}{2}$.

5. At a distance a from a tower on a horizontal plane the angle of elevation of its summit is the complement of that of a flag-staff upon it; show that the length of the flag-staff is $2a\cot 2\alpha$.

6. A building is three stories high, and from the opposite side of the street the angle of elevation of the roof is double, and that of the second story is the complement of the elevation of the first story. Show that the width of the street is a mean proportional between half the height of the roof and the difference of the heights of the first and second stories.

7. On the bank of a river there is a column 200 feet high

supporting a statue 30 feet high. To an observer on the opposite bank the statue subtends the same angle as a man six feet high standing at the base of the column. Find the breadth of the river.

8. Two railways intersect at an angle of 35° 20′; from the point of intersection two trains start together, one at the rate of 30 miles per hour, and at the end of $2\frac{1}{2}$ hours they are 50 miles apart; find the rate of the second train.

9. A sphere, radius r, on the top of a pole h feet high, subtends an angle of 2α at a point in the horizontal plane from which the elevation of the centre is β; show that the height of the pole is r ($\sin\beta \operatorname{cosec}\alpha - 1$).

10. The tops of three chimneys are in a horizontal line and at equal distances, c, from each other. From a point of obsevation on the horizontal plane below their angles of elevation are α, β, γ; show that their height is

$$\sqrt{\frac{2c^2}{\cot^2\alpha - 2\cot^2\beta + \cot^2\gamma}}.$$

11. A and B are two inaccessible points on a horizontal plane and C, D are two stations at each of which AB is observed to subtend an angle of 30°. AD subtends at C 19° 15,′ and AC subtends at D 40° 45′. Show that $CD = AB\sqrt{3}$.

12. From a station, A, at the foot of an inclined plane, AB, leading up to a mountain-side, BC, the angle of elevation of C is 60°. The inclination of AB is 30°, its length is a, and the angle ABC is 135°; show that the height of C is $\frac{a}{2}(3+\sqrt{3})$.

13. The angular elevation of the top of a steeple at a place due south of it is 30°, at a place due west of it the elevation is 18°, and the distance between the stations is a; show that its height is $\dfrac{a}{\sqrt{8+2\sqrt{5}}}$.

14. The angles of elevation of the top of a tower which leans to the north from two stations at distances a and b to the south, are α and β; show that the inclination of the tower is

$$\cot^{-1}\frac{b\cot\alpha - a\cot\beta}{b-a}.$$

15. The angles of elevation of an object at three horizontal stations A, B, C, lying in a vertical plane passing through it, are as $1:2:3$; if $AB=a$, $BC=b$, show that its height is

$$\frac{a}{2b}\sqrt{(a+b)(3b-a)}.$$

16. ABC is a horizontal line, CDE a vertical line, and DE subtends at A and B the same angle, α. If $AC=a$, $BC=b$, show that $DE=(a+b)\tan\alpha$.

17. ABC is a triangle right-angled at C, and the side $BC=a$; if the angles of elevation of an object at A from B and C are 15° and 45°, show that its height is $\frac{a}{2}(3^{\frac{1}{4}}-3^{-\frac{1}{4}})$.

18. An object $2b$ feet high, placed on the top of a tower, subtends an angle α at a place whose horizontal distance from the foot of the tower is b feet; show that the height of the tower is $b\{\sqrt{2}\cot\alpha - 1\}$.

19. The angles of elevation of a tower from three points A, B, C, in a straight line are observed to be α, β, γ, respectively. If $BC=a$, $AB=c$, show that the square of the height of the tower is

$$\frac{abc}{a\cot^2\alpha - b\cot^2\beta + c\cot^2\gamma}.$$

20. A tower standing on a horizontal plane is surrounded by a moat as wide as the tower is high. A person on the top of another tower whose height is a, and distance from the moat c, observes that the first tower subtends an angle of 45°; show that the height of the first tower is $\frac{a^2+c^2}{a-c}$.

21. A spherical balloon subtends angles α, β, γ at three points A, B, C, in a straight line. If $AB=a$, $BC=b$, show that its radius is

$$\sqrt{\frac{ab\,(a+b)}{a\ \mathrm{cosec}^2\frac{\gamma}{2}-(a+b)\,\mathrm{cosec}^2\frac{\beta}{2}+b\,\mathrm{cosec}^2\frac{\alpha}{2}}}.$$

22. Three points, A, B, C, are situated that so $AB=63$, $AC=44$, $BC=76$. From a point, P, in the same plane the sides AC and BC subtend angles of $20^\circ\ 10'$ and $30^\circ\ 20'$; find the distance of P from A, B and C.

23. In the preceding example if the angles subtended by AB and BC are $89^\circ\ 15'$ and $130^\circ\ 45'$ respectively; find the distance of P from A, B and C.

24. From a point on a hillside of constant inclination the altitude of the highest point of an obelisk at the top of a hill is observed to the α: a feet nearer the top of the hill it is β. Show that if θ be the inclination of the hill, the height of the obelisk is

$$\frac{a\sin(\alpha-\theta)\sin(\beta-\theta)}{\sin(\beta-\alpha)\cos\theta}.$$

25. From each of two stations on a horizontal plane, at a distance, c, from each other, a pillar on a distant hill in the vertical plane passing through the stations subtends the same angle, and the angles of elevation of the top of the pillar at the stations are α and β. Show that the length of the pillar is

$$\frac{c\cos(\beta+\alpha)}{\sin(\beta-\alpha)}.$$

26. A vertical stick casts a shadow of length b from a lamp upon a horizontal plane. The horizontal and vertical distances of the bottom of the stick from its shadow are a and c respectively. If the stick subtend equal angles at the two ends of the shadow, show that the height of the lamp is $\dfrac{abc}{c^2-a^2}$.

CHAPTER XV.

CIRCULAR MEASURE AND RATIOS OF ANGLES.

217. We now give a few of the simpler propositions which show the relation between the circular measure of angles and their trigonometrical ratios. The reasoning is based upon the two assumptions of Art. 31, which the reader should carefully review before proceeding with the subject.

218. *If θ be the circular measure of an arc in the first quadrant, sin θ, θ, tan θ, are in ascending order of magnitude.*

In the figure of Art. 33, let acb, ACB, be sides of regular polygons of n sides inscribed in, and described about, a circle of a unit radius. Denote the angle BOC by θ; then

$$cb = \sin\theta,\ Cb = \theta,\ CB = \tan\theta.$$

Now the perimeters of the inscribed polygon, the circle, and the described polygon are in order of magnitude, and therefore their $\left(\frac{1}{2n}\right)^{th}$ parts, viz., cb, Cb, CB are in order of magnitude, which proves the proposition.

219. *If θ is the circular measure of an angle, the limit of $\frac{\theta}{\sin\theta}$ when θ is indefinitely diminished is unity.*

We have $\quad \sin\theta,\ \theta,\ \tan\theta \quad$ in order of magnitude. Art. 218.
Divide each by $\sin\theta$.

Then $$1,\ \frac{\theta}{\sin\theta},\ \frac{1}{\cos\theta}$$ are in order of magnitude.

When θ is indefinitely diminished, $\cos\theta = 1$.

Hence $\dfrac{\theta}{\sin\theta}$, which always lies between 1 and $\dfrac{1}{\cos\theta}$, must also become a unit when $\theta=0$.

Cor. 1.—We have $\dfrac{\theta}{\tan\theta}=\dfrac{\theta}{\sin\theta}\times\cos\theta=1$ when $\theta=0$, since $\dfrac{\theta}{\sin\theta}$ and $\cos\theta$, each become 1 when $\theta=0$.

Cor. 2.—The reciprocals of $\dfrac{\theta}{\sin\theta}$ and $\dfrac{\theta}{\tan\theta}$, viz., $\dfrac{\sin\theta}{\theta}$ and $\dfrac{\tan\theta}{\theta}$ each become a unit when $\theta=0$.

Cor. 3.—The limit of $n\sin\dfrac{\theta}{n}$ when n is indefinitely great is θ.

For $n\sin\dfrac{\theta}{n}=\dfrac{\theta\sin\dfrac{\theta}{n}}{\dfrac{\theta}{n}}=\theta$, since $\dfrac{\theta}{n}$ is indefinitely small when n is indefinitely great, and consequently the ratios of $\sin\dfrac{\theta}{n}$ to $\dfrac{\theta}{n}$ is unity.

Cor. 4.—The limit of $\dfrac{\sin n^\circ}{n}$ is $\dfrac{\pi}{180}$. For if θ is the circular measure of n°, we have $\theta=\dfrac{n\pi}{180}$, or $n=\dfrac{180\,\theta}{\pi}$.

Then $\dfrac{\sin n^\circ}{n}=\dfrac{\sin\theta}{\dfrac{180\,\theta}{\pi}}=\dfrac{\sin\theta}{\theta}\cdot\dfrac{\pi}{180}=\dfrac{\pi}{180}$, since $\dfrac{\sin\theta}{\theta}=1$.

Cor. 5.—The limit of the ratio of the sine of an angle to the angle is the circular measure of the unit of angular measurement employed.

220. The proposition of the preceding article is very important. We give another proof founded upon a different principle which will be found instructive. In the figure of Art. 33, the angle BOC is $\frac{\pi}{n}$, where n is the number of the sides; denote this angle by θ. By sufficiently increasing n the angle $\frac{\pi}{n}$, or θ, may be made indefinitely small, whilst the perimeter of the polygon becomes ultimately equal to the circumference of the circle. We have, therefore,

$$\frac{\sin\theta}{\theta} = \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} = \frac{2nr\sin\frac{\pi}{n}}{2\pi r} = \frac{\text{perimeter of polygon}}{\text{circum. of circle}} = 1.$$

Similarly from the exterior polygon we may prove $\frac{\tan\theta}{\theta} = 1$.

221. *If θ is the circular measure of a positive angle less than a right angle, $\sin\theta$ is greater than $\theta - \frac{\theta^3}{4}$.*

We have

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2} = 2\tan\frac{\theta}{2}\left(1 - \sin^2\frac{\theta}{2}\right)$$

But $\tan\frac{\theta}{2} > \frac{\theta}{2}$, and $\sin\frac{\theta}{2} < \frac{\theta}{2}$. Art. 218.

Substituting these values in the above we get

$$\sin\theta > 2\,.\,\frac{\theta}{2}\left(1 - \frac{\theta^2}{4}\right)$$

which gives $\sin\theta > \theta - \frac{\theta^3}{4}$.

Cor.—Writing $\frac{\theta}{2}$ for θ we get $\sin\frac{\theta}{2} > \frac{\theta}{2} - \frac{\theta^3}{32}$.

222. *If θ is the circular measure of a positive angle less than a right angle, $\cos\theta$ is greater than*

$$1-\frac{\theta^2}{2},\ \textit{but less than}\ 1-\frac{\theta^2}{2}+\frac{\theta^4}{16}.$$

We have $\cos\theta=1-2\sin^2\frac{\theta}{2}>1-2\left(\frac{\theta}{2}\right)^2>1-\frac{\theta^2}{2}.$

Also
$$\cos\theta=1-2\sin^2\frac{\theta}{2}<1-2\left(\frac{\theta}{2}-\frac{\theta^3}{32}\right)^2$$
$$<1-\frac{\theta^2}{2}+\frac{\theta^4}{16}-2\left(\frac{\theta^3}{32}\right)^2$$
$$<1-\frac{\theta^2}{2}+\frac{\theta^4}{16},$$

Hence $\cos\theta>1-\frac{\theta^2}{2}$, but $<1-\frac{\theta^2}{2}+\frac{\theta^4}{16}.$

223. *To find the sine and the cosine of 10′.*

Let θ denote the circular measure of 10′.

Then
$$\theta=\frac{10\pi}{180\times 60}=.00290888,$$
and
$$\frac{\theta^3}{4}=.00000000601.$$

Hence θ and $\theta-\frac{\theta^3}{4}$ agree as far as eight decimal places, and since $\sin\theta$ lies between these values we have $\sin 10'=.00290888$ correct to eight places of decimals. Art. 221.

Similarly from Art. 222, $\cos 10'=.99999576$ to the same degree of approximation.

Cor. 1.—The sines of all angles less than 10′ are equal to their circular measures as far as eight places of decimals.

Cor. 2.—If n denote any number of seconds less than 600 (or 10′), then $\sin n''=n\sin 1''$ to eight places of decimals.

Cor. 3.—If a be not greater than $10'$ and $2 \cos a = 2 - k$, then

$$k = 2(1 - \cos a) = 4 \sin^2 \frac{a}{2} = \left(2 \sin \frac{a}{2}\right)^2 = \sin^2 a,$$

as far as eight decimal places.

224. *To find the sines of a series of angles which are multiples of 10′.*

If in the identity

$$\sin(n+1)a + \sin(n-1)a = 2 \sin na \cos a$$

we put $$2 \cos a = 2 - k,$$

we get $$\sin(n+1)a = 2 \sin na - \sin(n-1)a - k \sin na.$$

Let $a = 10'$ and for n write 1, 2, 3, etc., in succession,

Thus,

$$\sin 20' = 2 \sin 10' - k \sin 10'$$
$$\sin 30' = 2 \sin 20' - \sin 10' - k \sin 20'$$
$$\sin 40' = 2 \sin 30' - \sin 20' - k \sin 30', \text{ etc.}$$

These equations give the values of sin 20′, sin 30′, etc., in succession. We give a few steps of the work, which the student can easily continue, and compare his results with those given in the tables.

$\sin 10' =$	.00290888.	$k \sin 10' =$	.00000002.
$\sin 20' =$	581774.	$k \sin 20' =$	5.
$\sin 30' =$	872656.	$k \sin 30' =$	7.
$\sin 40' =$	1163531.	$k \sin 40' =$	10.
$\sin 50' =$	1454396.	$k \sin 50' =$	12.
$\sin 60' =$	1745249, etc.	$k \sin 60' =$	15, etc.

From Art. 223 (Cor. 3), $k = .00000846$. Each result as it is obtained is multiplied by k, and the product placed in the second column. It is then doubled, the previous result subtracted, and from this is taken the last product in the second column; this last remainder is the next result required.

225. From the identity

$$\cos (n+1) a + \cos (n-1) a = 2 \cos na \cos a$$
$$= 2 \cos na - k \cos na,$$

we get as before

$$\cos (n+1) a = 2 \cos na - \cos (n-1) a - k \cos na.$$

Thus, $\cos 20' = 2 \cos 10' - k \cos 10'.$
$\cos 30' = 2 \cos 20' - \cos 10' - k \cos 20'.$
$\cos 40' = 2 \cos 30' - \cos 20' - k \cos 30'$, etc.

The numerical values may be calculated as in the previous case.

226. In the process of Art. 224 the results are only approximations. In the beginning they are true to eight decimal places, but the errors accumulate in the course of the work, so as to affect the eighth or lower orders of decimals. We meet this difficulty by calculating independently, as in Chapter IV., the ratios for particular angles. Thus, from the known values of the ratios for 15° and 18° we can easily find those for 3° to any required degree of accuracy. From the use of the formulæ for the sum and difference, together with those for the bisection of angles we can correct our work at as many points as we please.

227. When the sines and cosines have been calculated up to 30° they may be continued up to 45° by the following formulæ which require subtraction only.

$$\sin (30° + a) = \cos a - \sin (30° - a)$$
$$\cos (30° + a) = \cos (30° - a) - \sin a.$$

Thus, if we put $a = 10'$ $20'$, etc., in succession we get,

$$\sin 30° \, 10' = \cos 10' - \sin 29° \, 50',$$
$$\cos 30° \, 10' = \cos 29° \, 50' - \sin 10', \text{ etc.}$$

When the sines and cosines have been calculated up to 45° the remainders are known from the relation $\sin (45° + a) = \cos (45° - a)$, etc. The values of the other ratios are easily ob-

tained from those of the sine and cosine. When the tables have been completed their accuracy may be tested in various ways. The equation

$$\sin(60^\circ + a) - \sin(60^\circ - a) = \sin a,$$

and the equations of example 51 and 52, page 124, are frequently used for this purpose, and are called formulæ of verification.

228. The leading principles of this chapter are conveniently expressed, thus:

1. When θ is small, $\sin\theta = \theta$, and $\tan\theta = \theta$ approximately.

2. When θ is indefinitely diminished, the ratios $\frac{\sin\theta}{\theta}$, $\frac{\tan\theta}{\theta}$ become unity exactly.

The first proposition is frequently used in practical problems in which only approximate measurements can be made. The second is used in proving theorems in which absolute accuracy is required. The remaining articles of this chapter illustrate these distinctions.

229. *To find the value of* $\cos\frac{\theta}{2}.\cos\frac{\theta}{4}.\cos\frac{\theta}{8}\ldots\ldots\cos\frac{\theta}{2^n}$ *when* n *is indefinitely increased and* θ *is the circular measure of any finite angle.*

We have $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2^2\sin\frac{\theta}{4}\cos\frac{\theta}{4}\cos\frac{\theta}{2} = \text{etc.}$

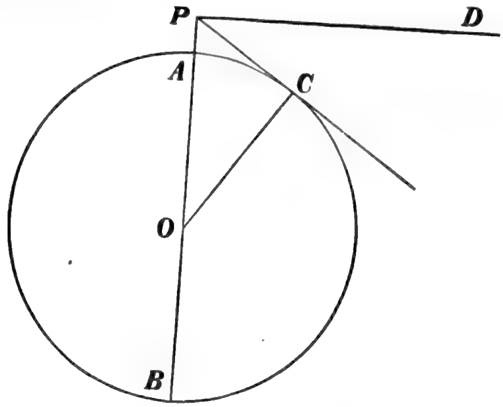
Thus $$\sin\theta = 2^n\sin\frac{\theta}{2^n}\left(\cos\frac{\theta}{2}.\cos\frac{\theta}{2^2}.\cos\frac{\theta}{2^3}\ldots\ldots\cos\frac{\theta}{2^n}\right).$$

But when n becomes indefinitely great so also does 2^n, and consequently $2^n\sin\frac{\theta}{2^n} = \theta$. Substituting this value we get

$$\frac{\sin\theta}{\theta} = \cos\frac{\theta}{2}.\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\ldots\ldots\cos\frac{\theta}{2^n} \text{ when } n \text{ is indefinitely great.}$$

230. The circle marking the limit of vision on the surface of the earth is called the **horizon,** and the angle of depression of any point in this circle below the horizontal is called the **dip of the horizon.**

231. *To find the distance and dip of the horizon from a given point of observation.*



Let the circle ACB denote the earth, P, a point of observation above the surface, PC a tangent, PD the horizontal, PO the vertical; then C is a point in the horizon, the arc AC is the distance, and the angle $CPD =$ angle COA is the dip of the horizon.

Then rectangle $\qquad BP.PA = PC^2. \qquad$ Euc. III., 36.

Now AP will usually be much less than the thousandth part of AB, and the angle AOC will be very small; consequently $BA = BP$, and arc $AC = PC$ approximately. Denote the height of AP in feet by h, the distance AC in miles by d, and assume $BP = 8000$ miles.

Then approximately $\quad d^2 = 8000 \times \dfrac{h}{5280} = \dfrac{3h}{2}$ nearly.

Also, $\quad$ angle $COA = \tan^{-1}\dfrac{PC}{OC} = \tan^{-1}\left(\dfrac{1}{8000}\sqrt{\dfrac{3h}{2}}\right).$

232. *To show that* $\sin\theta > \theta - \frac{\theta^3}{6}$ *where* θ *is the circular measure of an acute angle.*

We have $\cos\frac{\theta}{2} > 1 - \frac{\theta^2}{2^3}$, $\cos\frac{\theta}{4} > 1 - \frac{\theta^2}{2^5}$, etc. Art. 222.

Therefore $\frac{\sin\theta}{\theta} > \left(1 - \frac{\theta^2}{2^3}\right)\left(1 - \frac{\theta^2}{2^5}\right)\left(1 - \frac{\theta^2}{2^7}\right)\ldots.$ Art. 231.

$$> 1 - \frac{\theta^2}{8}\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \ldots.\right) + \ldots.$$

$$> 1 - \frac{\theta^2}{6},$$

or $$\sin\theta > \theta - \frac{\theta^3}{6}.$$

EXERCISE XXIX.

1. Show that the limit of the ratio of $\sin k\theta$ to k, when k is indefinitely diminished and θ is finite is θ.

2. Show that $\tan\theta$ is greater than $\theta + \frac{\theta^3}{3}$.

3. Show that $\cot\theta > \frac{1}{\theta} - \frac{\theta}{2}$ but $< \frac{1}{\theta}$.

4. Express $\tan\theta$ in terms of $\tan\frac{\theta}{2}$, and thence show that $\tan\theta > \theta + \frac{\theta^3}{4} + \frac{\theta^5}{16} + \frac{\theta^7}{64} + \ldots.$

5. Show that $\tan\theta < \frac{2\theta}{2 - \theta^2}$ but $> \frac{4\theta}{4 - \theta^2}$.

6. From Arts. 218 and 221, prove $\cot\theta < \frac{1}{\theta} - \frac{\theta}{4}$.

7. The chord of an arc subtending an angle θ at the centre of the circle is $2r\sin\frac{\theta}{2}$ and the chord of half the arc is $2r\sin\frac{\theta}{4}$.

8. From Art. 232 prove the following rule for finding approximately the length of the arc of a circle. From eight times the

chord of half the arc subtract the chord of the whole arc; one-third of the remainder will be the length of the arc.

9. Find the dip of the horizon from the top of a mountain $1\frac{3}{4}$ miles high, the radius of the earth being 4000 miles.

10. Having given that two points, each 10 feet above the earth's surface, cease to be visible from each other over still water at a distance of eight miles; find the earth's diameter.

11. If θ be the dip of the horizon from the top of a mountain, and R the radius of the earth, show that the height is approximately $\frac{1}{2} R \tan^2 \theta$.

12. Given that the moon subtends at the earth an angle of half a degree, find the distance at which a circular plate of six inches in diameter must be placed so as just to conceal it.

13. Find the value of $\dfrac{\sin a\theta}{\sin b\theta}$, when θ is indefinitely diminished.

14. If θ be the circular measure of a small angle of n'', prove that approximately

$$\log n + \log \frac{\sin \theta}{\theta} = L \sin n'' - L \sin 1''.$$

15. When θ is small, prove that approximately

$$(1)\ \log \sin \theta = \log \theta + \tfrac{1}{3} \log \cos \theta.$$
$$(2)\ \log \tan \theta = \log \theta - \tfrac{2}{3} \log \cos \theta.$$

16. If p be the perimeter of a regular polygon inscribed in a circle whose diameter is 1, and θ half the angle subtended at the centre by one of the sides, then

$$\pi = p \sec \frac{\theta}{2} \sec \frac{\theta}{4} \sec \frac{\theta}{8} \ldots \textit{ad. inf.}$$

EXAMINATION PAPERS.

PAPER I.

1. Explain the difference between the trigonometrical and the geometrical line.

2. The minute hand of a clock is $3\frac{1}{2}$ inches long; in what length of time will its extremity move $2\frac{3}{4}$ inches?

3. Show that $\sin^4 A + \sin^2 A \cos^2 A + \cos^4 A = 1 - \sin^2 A \cos^2 A$.

4. Given $\tan \theta = \sqrt{2}$, $\sin \theta$; find three different values for θ

5. In a triangle right-angled at C, show that $a^2 \tan B = b^2 \tan A =$ twice area of triangle.

6. Prove $\dfrac{1 + \tan A}{1 - \tan A} + \dfrac{1 - \tan A}{1 + \tan A} = 2 \sec 2A$.

7. The sides of a triangle are 14, 15, 16; find the cosine of the least angle.

8. Express $\log \cos 45°$ in terms of $\log 2$.

9. Show that the radius of the circle circumscribing a triangle right-angled at C is $\dfrac{a}{2} \sec B$.

10. Show that the least value of $\sin^{-1} \frac{3}{5} + \sin^{-1} \frac{4}{5}$ is $\dfrac{\pi}{2}$.

PAPER II.

1. Explain the difference between the trigonometrical and the geometrical angle.

2. The angles of a triangle are in arithmetical progression, and the circular measure of their common difference is $\dfrac{\pi}{9}$; express the angles in degrees.

3. Prove $(\sin A + \cos A)^3 + (\sin A - \cos A)^3 = 2 \sin A\,(3 - 2\sin^2 A)$.

4. Given $\sin A + \operatorname{cosec} A = 2$; find $\cos A$.

5. Show that the area of a triangle right-angled at C is $\frac{1}{2}(a^2 + b^2) \cos A \cos B$.

6. Solve the equation $\dfrac{1 + \tan A}{1 - \tan A} = 2 \tan 2A$.

7. The sides of a triangle are 5, 12, 13; find the radius of the inscribed circle.

8. In a triangle $\log \sin C = 0$, and $\log \sin B = \frac{1}{2} \log 3 - \log 2$; find all the angles.

9. The sides of a triangle are 7, 24, 25; find the radius of the largest escribed circle.

10. If $\sin^{-1} m + \sin^{-1} n = \dfrac{\pi}{2}$; show that $\sin^{-1} m = \cos^{-1} n$.

PAPER III.

1. State accurately the meaning of π as used in trigonometry.

2. In a quadrilateral $ABCD$, the angle $A = 30°$, $B = 60^g$, $C = \frac{2}{3}\pi$; find the number of degrees which must be taken from D and added to A, so that the figure may be inscribed in a circle.

3. Prove $\cot^2 A \left(\dfrac{\sec A - 1}{1 + \sin A}\right) + \sec^2 A \left(\dfrac{\sin A - 1}{1 + \sec A}\right) = 0$.

4. Given $8 \cos^4 \theta - 8 \cos^2 \theta + 1 = 0$; find $\cos 2\theta$ and $\cos 4\theta$.

5. In any triangle show that $a \cos B + b \cos A$ is positive

6. Prove $2 \cot 2\theta\,(\tan \theta + \cot \theta) = \operatorname{cosec}^2 \theta - \sec^2 \theta$.

7. The sides of a triangle are 6, $6 + \sqrt{2}$, $6 - \sqrt{2}$; find its area and the sine of the medium angle.

8. Two adjacent sides of a parallelogram are of lengths 25 and 30, and the angle between them 60°; find the lengths of the diagonals.

9. Prove $\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{3}=\tan^{-1}1$. Illustrate geometrically.

10. Prove $\log\cos A+\log\cos(60^\circ+A)+\log\cos(60^\circ-A)+\log 4=\log\cos 3A$.

PAPER IV.

1. Name the quadrants in which the several trigonometrical ratios are respectively positive.

2 The perimeter of a sector of a circle is equal to half the circumference of a circle; how many degrees in the angle of the sector?

3. Simplify $(a+b)\cos 180^\circ+(a-b)\sin 90^\circ+2b\tan 45^\circ$.

4. In any triangle if $A=60^\circ$, then $b+c=\sqrt{a^2+3bc}$.

5. When the altitude of the sun is $22^\circ\ 30'$, find the length of the longest shadow that can be cast upon a horizontal plane by a rod 12 feet in length.

6. Show that the area of a parallelogram is equal to the continued product of its diagonals and the sine of the angle between them.

7. Solve the equation $\cot^2\theta+\tan^2\theta=\frac{10}{3}$.

8. In a triangle right-angled at C, prove

$$\tan^{-1}\frac{a}{b+c}+\tan^{-1}\frac{b}{a+c}=\frac{\pi}{4}.$$

9. Given $3^{2x}5^{x-1}=2^{3x+1}$; find x.

10. Prove that in any triangle

$$\tan 2A+\tan 2B+\tan 2C=\tan 2A\tan 2B\tan 2C.$$

PAPER V.

1. State the independent relations which exist between the six trigonometrical ratios of the angles of a triangle. Which of them require proof?

2. If $\tan 15^\circ = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, then $x = \frac{1}{4}\log_e 3$.

3. The distance between the extremities of the perpendiculars from any point in a circular arc on the radii through the extremities of the arc is constant.

4. Prove that the line which divides one of the angles of an equilateral triangle in the ratio $3:1$ divides the opposite side in the ratio $\sqrt{3}+1:2$.

5. In any triangle if a, b, c are in harmonical progression, $\sin^2 A$, $\sin^2 B$, $\sin^2 C$ are in arithmetical progression.

6. Given $\sin l = \dfrac{a-b}{a+b}$, $\sin m = \dfrac{b-c}{b+c}$, $\sin n = \dfrac{c-a}{c+a}$, prove that $\sec^2 l + \sec^2 m + \sec^2 n = 2 \sec l . \sec m . \sec n + 1$.

7. If a, b, c, d are the sides of a quadrilateral taken in order and θ the angle between its diagonals, the area of the quadrilateral is $\frac{1}{4}(a^2 - b^2 + c^2 - d^2)\tan\theta$, the sides b, d being opposite the angle θ.

8. Eliminate x and y from the equations

$$\tan x + \tan y = a, \quad \cot x + \cot y = b, \quad x + y = c.$$

9. $ABCD$ is a rectangle, BO is a perpendicular on the diagonal AC; QOR, SOP are parallel to AB, BC, and θ is the angle BAC. Prove $OC = AO\tan^2\theta$, $OP = OS\tan^2\theta$, $OR = OQ\tan^2\theta$.

10. In the preceding example if $OP = p$, $OQ = q$, $AC = c$, express p and q in terms of c and θ, and thus show that

$$p^{\frac{2}{3}} + q^{\frac{2}{3}} = c^{\frac{2}{3}}.$$

PAPER VI.

1. Name the limits between which the trigonometrical ratios respectively lie.

2. In what length of time will the extremity of the minute hand of a clock move a distance equal to its own length?

3. Prove $\sec A + \tan A = \tan(45^\circ + \frac{1}{2}A)$, and $\tan\frac{A}{2} = \frac{\text{vers } A}{\sin A}$.

4. Solve the equation $\cos^4\theta - \sin^4\theta = \frac{\sqrt{6} - \sqrt{2}}{4}$.

5. If in any triangle $\frac{a+c}{b+c} = \frac{b}{a-c}$, then $A = 120^\circ$.

6. Solve the equation $\tan A + \tan 2A = \tan 3A$.

7. Find the area of a pentagon each of whose diagonals is 10 inches.

8. If $h\cos\theta + k\sin\theta = 1$, and $l\cos\theta + m\sin\theta = 1$, prove that $(l-h)^2 + (m-k)^2 = (lh - mk)^2$.

9. If $\cot\phi = \sqrt{2} + 1$ and ϕ is an angle in the third quadrant, find $\sin\phi$ and $\cos\phi$.

10. A ring 12 inches in diameter is suspended from a point by six strings, each 10 inches long, attached to points in the circumference at equal intervals; find the angle between two consecutive strings and the angle each string makes with the vertical.

PAPER VII.

1. In a triangle right-angled at C, $\sin(45^\circ \pm \frac{1}{2}B) = \sqrt{\frac{c \pm b}{2c}}$.

2. Prove $\sin(A+B) : \sin A + \sin B :: \sin A - \sin B : \sin(A-B)$.

3. Prove $\sin A + \cot A + \sec A - \cos A - \tan A - \operatorname{cosec} A$
$= (\sin A - 1)(\cot A - 1)(\sec A - 1)$.

4. In any triangle $\tan\left(\frac{A}{2} + B\right) = \frac{c+b}{c-b}\tan\frac{A}{2}$.

5. The hands of a clock are 3 and 2 inches in length; find the first time after three o'clock when the extremities of the hands are two inches apart.

6. In any triangle

$$4 \text{ area} = \frac{a^2}{\tan A} + \frac{b^2}{\tan B} + \frac{c^2}{\tan C} = \frac{a^2 + b^2 + c^2}{\cot A + \cot B + \cot C}.$$

7. Express the area of a triangle in terms of a side and the two adjacent angles.

8. In any triangle $\sin(A - B)\sin(B - C)\sin(C - A)$

$$= \left(\sin 2A - \frac{a}{c}\sin B\right)\left(\sin 2B - \frac{b}{a}\sin C\right)\left(\sin 2C - \frac{c}{b}\sin A\right).$$

9. Find $\log \dfrac{\sqrt{21.43} \times \sqrt[3]{.27037}}{(46.325)^5}$ and $\log \dfrac{105.52\sqrt{\sin 15^\circ 12'}}{(\tan 26^\circ 13')^2}$.

10. If $\cos x = \dfrac{a}{b+c}$, $\cos y = \dfrac{b}{c+a}$, $\cos z = \dfrac{c}{a+b}$,

then $$\tan^2 \frac{x}{2} + \tan^2 \frac{y}{2} + \tan^2 \frac{z}{2} = 1,$$

and $$\tan \frac{x}{2} \tan \frac{y}{2} \tan \frac{z}{2} = \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}.$$

PAPER VIII.

1. Can an angle be completely determined from the known value of one of its ratios? Explain clearly.

2. A cube the length of whose edge is a is placed with its diagonal vertical; find the height of each of the corners above the point of support.

3. If x, y, z are the perpendiculars from any point within a triangle on the sides a, b, c; then $ax + by + cz = 2S$. Examine the result when the point is taken outside the triangle.

4. If R be the radius of the circle circumscribing a triangle, show that the product of the perpendiculars from the angles on the opposite sides is $\dfrac{a^2 b^2 c^2}{8R^3}$.

5. In any triangle, show that $\sin A \sin B \sin C$
$= \sin A \cos B \cos C + \sin B \cos C \cos A + \sin C \cos A \cos B$.

6. From $m \sin 2\theta = n \sin \theta$, $p \cos 2\theta = q \cos \theta$, obtain an equation independent of θ.

7. In the triangle ABC the straight line joining A to the middle point of BC is at right angles to AC; show that

$$\cos A \cos C = \frac{2(c^2 - a^2)}{3ac}.$$

8. If AD, BE be drawn bisecting the angles of a triangle, and if r_1, r_2 are the radii of the circles inscribed in ABD, ABE, and S the area of ABC; then

$$\frac{1}{r_2} - \frac{1}{r_1} = \frac{1}{S}\left(a \cos \frac{B}{2} - b \cos \frac{A}{2}\right).$$

9. Solve the equations,

$$\sin^2(\theta + \phi) - \sin^2(\theta - \phi) = \frac{\sqrt{3}}{2},$$

$$\operatorname{cosec} 2\theta + \cos 2\phi = \frac{2}{\sqrt{3}}.$$

10. Prove

$$\log \cos \frac{x}{2} + \log \cos \frac{x}{4} + \log \cos \frac{x}{8} + \dots \; ad.\ inf. = \log \sin x - \log x.$$

PAPER IX.

1. Show that when the tangent of half an angle of a triangle is known all the other ratios of that angle may be determined without ambiguity.

2. In a triangle $c = (a + b)\dfrac{\sin \frac{1}{2} C}{\cos \phi}$, if $\tan \phi = \dfrac{a - b}{a + b} \cot \dfrac{C}{2}$.

3. The distances of the centres of the escribed circles of a triangle from the centre of the inscribed circle are as

$$\frac{1}{a}(r_1 - r) \cos \frac{A}{2} : \frac{1}{b}(r_2 - r) \cos \frac{B}{2} : \frac{1}{c}(r_3 - r) \cos \frac{C}{2}.$$

4. In the ambiguous case a, b, A, being given, if c_1, c_2 are the third sides of the two triangles, show that the distance between the centres of their circumscribing circles is $\dfrac{c_2 - c_1}{2 \sin A}$.

5. The sides of a parallelogram are a and b, and the angle between them is θ; show that the tangent of the angle between its diagonals is $\dfrac{2ab \sin \theta}{a^2 - b^2}$.

6. Eliminate a, b, θ from the equations

$$x = b \cos(\theta - \phi) - c = a \cos \theta + c.$$
$$y = b \sin(\theta - \phi) \quad = a \sin \theta.$$

7. If D, E, F are the feet of the perpendiculars from A, B, C, upon the opposite sides of the triangle ABC, the diameters of the circumscribing circles of the triangles AEF, BDF, CDE, are $a \cot A$, $b \cot B$, $c \cot C$ respectively.

8. If $\tan(A+B+C) = 0$, then $A+B+C = n\pi$, and $\tan(2A+2B+2C) = 0$. Hence prove trigonometrically that if $x+y+z = xyz$, then

$$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{8xyz}{(1-x^2)(1-y^2)(1-z^2)}.$$

Extend the same principle to obtain other similar identities.

9. If $\sqrt{y^2+yz+z^2}$, $\sqrt{z^2+zx+x^2}$, $\sqrt{x^2+xy+y^2}$ are the sides of a triangle; then $\dfrac{\sqrt{3}}{4}(yz+zx+xy)$ is the area.

10. Three circles, two of which are equal touch each other, and a fourth circle lies between them touching each. The radius of the equal circles is r', that of the third circle is r, and θ is the angle between the lines joining the centres of the equal circles to to the centre of the third circle. Show that the radius of the fourth circle is

$$\frac{r(r'+r)\sin^2 \frac{1}{4}\theta}{r' \cos^2 \frac{1}{4}\theta - r \sin^2 \frac{1}{4}\theta}.$$

Examination Papers in Trigonometry usually furnish the Logarithms required in the solution of the given problems. We give three such papers, to render the student familiar with the form in which his work will be presented to him.

PAPER X.

TORONTO UNIVERSITY.

1. How many digits in the integral part of $(2{\cdot}5)^{20}$?

2. Simplify $\sqrt[4]{80} \times \sqrt[3]{2.7}$, $\sqrt[5]{-5} \times (18)^{-\frac{1}{5}}$.

3. Show that the logarithms of the trigonometrical ratios need not be entered for angles greater than 45°. Illustrate by using the last two values given below to find the values for other logarithmic ratios.

4. Adapt $\sin A - \tan \frac{1}{2} A$ to logarithmic computation, and find its logarithm when $A = 53° \, 6'$.

5. Given $B = 123° \, 40'$, $b = 100$, $c = 60$; find A and C.

6. Given $A = 112° \, 40'$, $b = 213.4$, $c = 213.4$; solve the triangle.

7. Given $a = 200$, $b = 77.4$, $C = 41° \, 50'$; find the area.

8. If $(\sin \theta + \cos \theta)^2 = 3 \sin \theta + \sin 2\theta$; find θ in degrees.

9. If $1 + \sin \theta = 2 \cos \frac{1}{2} \theta (\cos \frac{1}{2} \theta - \sin \frac{1}{2} \theta)$; find θ in degrees.

10. A person standing on one bank of a river observes that an object on the other bank has an angle of elevation of 45°, and on going back 150 feet the corresponding angle is 30°; find the breadth of the river.

11. A vertical rod 10 feet long casts a shadow 7.74 feet long on a horizontal plane; find the sun's altitude.

NUMBER.	LOG.	ANGLE.	LOG.	DIFF. FOR 60″
20000	301030	tan 52° 15′	10.11110	26
30000	47712	sin 56° 20′	9.92027	
41645	61956	sin 29° 57′ 30″	9.69842	
21340	32919	sin 41° 50′	9.82410	
17761	24946	sin 19° 28′	9.52278	
51623	71284	tan 26° 33′	9.69868	36
		cos 53° 6′	9.77845	32

PAPER XI.

TORONTO UNIVERSITY.

1. Multiply 501.26 by .399.

2. Find what per cent. .39342 is of 78.492.

3. Simplify $\{\sqrt[5]{.05} \times (36)^{-2}\} \div 5.4$.

4. Given $2^x = 399$; find x.

5. Find L sin 24° 45′ 15″, L cosec 24° 45′ 50″, and the angle whose L cos is 9.62204.

6. Given $a = 589.17$, $b = 195.75$, $C = 52°$; find A, B, c.

7. Given $A = 12° \ 43'$, $C = 90°$ and the area $= 1995$; find a, b, c, B.

8. Given $A = 109° \ 47'$, $a = 589.17$, $b = 78.492$; find B.

9. If l, m, n are the bisectors of the angles of a triangle ABC; S, the area, and $2s = a + b + c$,

then
$$\frac{al\,(2s-a)}{\cos\frac{1}{2}A} = \frac{bm\,(2s-b)}{\cos\frac{1}{2}B} = \frac{cn\,(2s-c)}{\cos\frac{1}{2}C},$$
and
$$\frac{lmn}{abc} = \frac{8sS}{(a+b)\,(b+c)\,(c+a)}.$$

10. If l, m, n are the bisectors of the opposite sides of a triangle, and $2s = l + m + n$, then its area is $\frac{4}{3}\sqrt{s(s-a)(s-b)(s-c)}$.

NUMBER.	LOG.	ANGLE.	LOG.
20000	30103	L cot 26°	10.31182
30000	47712	L sin 52°	9.89653
78492	89480	L sin 70° 13′	9.97358
39342	59486	L tan 12° 43′	9.35327
50126	70006	L tan 45° 47′	10.01188
39900	60097	L sin 24° 45′	9.62186
58917	77024	L sin 24° 46′	9.62213
49341	69319	L sin 7° 12′	9.09814

PAPER XII.

QUEEN'S, TRINITY AND VICTORIA.

1. The circular measure of the angle C is $\frac{7\pi}{18}$, and of the sum of A, B, C is $\frac{3\pi}{2}$; the number of grades in the difference of A and B is 40; find the number of degrees in each angle.

2. Define the trigonometrical functions.

Prove (1) $\cos A + \sin A = (1 + \tan A) \cos A$.

$$(2)\ \tan \frac{A}{2} = \frac{\sin 2A}{1 + \cos 2A} \cdot \frac{\cos A}{1 + \cos A}.$$

$$(3)\ y = \cot^{-1}\left(\frac{\sqrt{1 - x^2}}{x}\right), \text{ if } y = \sin^{-1} x.$$

3. Given $\sin a = .1$; find $\cos a$, $\tan a$, $\cot a$, $\sec a$, $\sin 2a$ and $\cos 2a$.

4. Calculate to three decimal places the value of the sine of an arc subtended by a chord whose length is $\frac{2}{3}$ of the diameter.

5. Prove the following identities:

$$(1)\ \frac{\sin 7x}{\sin x} - 2 \cos 2x - 2 \cos 4x - 2 \cos 6x = 1.$$

$$(2)\ \sin \gamma = \frac{\tan a}{\tan \beta}, \text{ if } \left(\frac{\sin a}{\sin \beta}\right)^2 + (\cos a \cos \gamma)^2 = 1.$$

$$(3)\ \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} \text{ if } \pi = A + B + C.$$

6. Prove geometrically,

(1) $\cos (x + y) = \cos x \cos y - \sin x \sin y$, when $x + y > \frac{\pi}{2}$.

(2) $\cos x + 1 = 2 \cos^2 \frac{x}{2}$.

(3) From (1) deduce

$$\cos x + \cos y = 2 \cos \tfrac{1}{2} (x + y) \cos \tfrac{1}{2} (x - y).$$

7. Make convenient for logarithmic computation,

(1) $1 - \tan x \tan y$. (2) $\frac{\tan x + \tan y}{\cot x + \cot y}$. (3) $\frac{1 - \cos 2x}{1 + \cos 2x}$.

8. Prove, using the ordinary notation, that in any triangle

(1) $\dfrac{a+b}{a-b}=\dfrac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$. (2) $\sin\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{bc}}$.

(3) $b\cot\dfrac{A}{2}+c\cot\dfrac{B}{2}+a\cot\dfrac{C}{2}=c\cot\dfrac{A}{2}+a\cot\dfrac{B}{2}+b\cot\dfrac{C}{2}$.

(4) Area of triangle $=\dfrac{ab+bc+ca}{2(\operatorname{cosec}A+\operatorname{cosec}B+\operatorname{cosec}C)}$.

(5) If in (1), $A-B=90^\circ$ and $B=C$, describe the triangle and find its angles.

9. Show how to solve the triangle when a, b, A are given, and discuss the ambiguity in full.

If Δ and δ be the areas of the two triangles, prove

$$\Delta^2+\delta^2-2\Delta\delta\cos 2A=\frac{a^2}{b^2}(\Delta+\delta)^2.$$

10. Deduce the expressions for the radii of the inscribed and circumscribed circles of a triangle whose sides are known. Prove that the area of the triangle of which the vertices are the points of contact of the inscribed circle is to the area of the original triangle as the radius of the inscribed is to the diameter of the circumscribed circle.

11. Find the angles in the following triangles:

(1) $a=5082$, $b=4522$, $C=45^\circ\ 7'$.
(2) $a=343$, $b=833$, $B=50^\circ$.

12. From a station on the top of a hill, three towers on a horizontal plane are found to subtend equal angles at the eye of an observer, and the angles of depression of their bases are α, β, γ. Prove that n, p, q, being the height of the towers,

$$\frac{\sin(\beta-\gamma)}{n\sin\alpha}+\frac{\sin(\gamma-\alpha)}{p\sin\beta}+\frac{\sin(\alpha-\beta)}{q\sin\gamma}=0.$$

NUMBER.	LOG.	ANGLE.	LOG.	DIFF. FOR 60″.
7000	8450980	cot 22° 23′	10.3817047	3566
2000	3010300	tan 7° 59′	9.1468849	9176
1190	0755470	sin 18° 23′	9.4988245	3800
		sin 25°	9.6259483	
		cos 25°	9.9572757	

APPENDIX.

MATHEMATICAL TABLES

CONSISTING OF

I. LOGARITHMS OF NUMBERS.

II. NATURAL SINES, COSINES, TANGENTS AND COTANGENTS.

III. LOGARITHMS, SINES, COSINES, TANGENTS AND COTANGENTS.

IV. NUMBERS OFTEN USED IN CALCULATIONS.

No.	Log.	No.	Log.	No.	Log	No.	Log.	No.	Log.
100	00000	145	16137	190	27875	235	37107	280	44716
1	432	6	435	1	28103	6	291	1	871
2	860	7	732	2	330	7	475	2	45025
3	01284	8	17026	3	556	8	658	3	179
4	703	9	319	4	780	9	840	4	332
105	02119	150	17609	195	29003	240	38021	285	45484
6	531	1	898	6	226	1	202	6	637
7	938	2	18184	7	447	2	382	7	788
8	03342	3	469	8	667	3	561	8	939
9	743	4	752	9	885	4	739	9	46090
110	04139	155	19033	200	30103	245	39917	290	46240
1	532	6	312	1	320	6	094	1	389
2	922	7	590	2	535	7	270	2	538
3	05308	8	866	3	750	8	445	3	687
4	690	9	20140	4	963	9	620	4	835
115	06070	160	20412	205	31175	250	39794	295	46982
6	446	1	683	6	387	1	967	6	47129
7	819	2	951	7	597	2	40140	7	276
8	07188	3	21219	8	806	3	312	8	422
9	555	4	484	9	32015	4	483	9	567
120	07918	165	21748	210	32222	255	40654	300	47712
1	08279	6	22011	1	428	6	824	1	857
2	636	7	272	2	634	7	993	2	48001
3	991	8	531	3	838	8	41162	3	144
4	09342	9	789	4	33041	9	330	4	287
125	09691	170	23045	215	33244	260	41497	305	48430
6	10037	1	300	6	446	1	664	6	572
7	380	2	553	7	646	2	830	7	714
8	721	3	805	8	846	3	996	8	855
9	11059	4	24055	9	34044	4	42160	9	996
130	11394	175	24304	220	34242	265	42325	310	49136
1	727	6	551	1	439	6	488	1	276
2	12057	7	797	2	635	7	651	2	415
3	385	8	25042	3	830	8	813	3	554
4	710	9	285	4	35025	9	975	4	693
135	13033	180	25527	225	35218	270	43136	315	49831
6	354	1	768	6	411	1	297	6	969
7	672	2	26007	7	602	2	457	7	50106
8	988	3	245	8	793	3	616	8	243
9	14301	4	482	9	984	4	775	9	379
140	14613	185	26717	230	36173	275	43933	320	50515
1	922	6	951	1	361	6	44091	1	651
2	15229	7	27184	2	549	7	248	2	786
3	534	8	416	3	736	8	404	3	920
4	836	9	646	4	922	9	560	4	51055

No.	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Log.
325	51188	370	56820	415	61805	460	66276	505	70329
6	322	1	937	6	909	1	370	6	415
7	455	2	57054	7	62014	2	464	7	501
8	587	3	171	8	118	3	558	8	586
9	720	5	287	9	221	4	652	9	672
330	51851	375	57403	420	62325	465	66745	510	70757
1	983	6	519	1	428	6	839	1	842
2	52114	7	634	2	531	7	932	2	927
3	244	8	749	3	634	8	67025	3	71012
4	375	9	864	4	737	9	117	4	096
335	52504	380	57978	425	62839	470	67210	515	71181
6	634	1	58093	6	941	1	302	6	265
7	763	2	206	7	63043	2	394	7	349
8	892	3	320	8	144	3	486	8	433
9	53020	4	433	9	246	4	578	9	517
340	53148	385	58546	430	63347	475	67669	520	71600
1	275	6	659	1	448	6	761	1	684
2	403	7	771	2	548	7	852	2	767
3	529	8	883	3	649	8	943	3	850
4	656	9	995	4	749	9	68034	4	933
345	53782	390	59106	435	63849	480	68124	525	72016
6	908	1	218	6	949	1	215	6	099
7	54033	2	329	7	64048	2	305	7	181
8	158	3	439	8	147	3	395	8	263
9	283	4	550	9	246	4	485	9	346
350	54407	395	59660	440	64345	485	68574	530	72428
1	531	6	770	1	444	6	664	1	509
2	654	7	879	2	542	7	753	2	591
3	777	8	988	3	640	8	842	3	673
4	900	9	60097	4	738	9	931	4	754
355	55023	400	60206	445	64836	490	69020	535	72835
6	145	1	314	6	933	1	108	6	916
7	267	2	423	7	65031	2	197	7	997
8	388	3	530	8	128	3	285	8	73078
9	509	4	638	9	225	4	373	9	159
360	55630	405	60746	450	65321	495	69461	540	73239
1	751	6	853	1	418	6	548	1	320
2	871	7	959	2	514	7	636	2	400
3	991	8	61066	3	610	8	723	3	480
4	56110	9	172	4	706	9	810	4	560
365	56229	410	61278	455	65801	500	69897	545	73640
6	348	1	384	6	896	1	984	6	719
7	467	2	490	7	992	2	70070	7	799
8	585	3	595	8	66087	3	157	8	878
9	703	4	700	9	181	4	243	9	957

No.	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Log.
550	74036	595	77452	640	80618	685	83569	730	86332
1	115	6	525	1	686	6	632	1	392
2	194	7	597	2	754	7	696	2	451
3	273	8	670	3	821	8	759	3	510
4	351	9	743	4	889	9	822	4	570
555	74429	600	77815	645	80956	690	83885	735	86629
6	507	1	887	6	81023	1	948	6	688
7	586	2	960	7	090	2	84011	7	747
8	663	3	78032	8	158	3	073	8	806
9	741	4	104	9	224	4	136	9	864
560	74819	605	78176	650	81291	695	84198	740	86923
1	896	6	247	1	358	6	261	1	982
2	974	7	319	2	425	7	323	2	87040
3	75051	8	390	3	491	8	385	3	099
4	128	9	462	4	558	9	448	4	157
565	75205	610	78533	655	81624	700	84510	745	87216
6	281	1	604	6	690	1	572	6	274
7	358	2	675	7	757	2	634	7	332
8	435	3	746	8	823	3	696	8	390
9	511	4	817	9	889	4	757	9	448
570	75587	615	78888	660	81954	705	84819	750	87506
1	664	6	958	1	82020	6	880	1	564
2	740	7	79029	2	086	7	942	2	622
3	815	8	099	3	151	8	85003	3	680
4	891	9	169	4	217	9	065	4	737
575	75967	620	79239	665	82282	710	85126	755	87795
6	76042	1	309	6	347	1	187	6	852
7	118	2	379	7	413	2	248	7	910
8	193	3	449	8	478	3	309	8	967
9	268	4	518	9	543	4	370	9	88024
580	76343	625	79588	670	82607	715	85431	760	88081
1	418	6	657	1	672	6	491	1	138
2	492	7	727	2	737	7	552	2	196
3	567	8	796	3	802	8	612	3	252
4	641	9	865	4	866	9	673	4	309
585	76716	630	79934	675	82930	720	85733	765	88366
6	790	1	80003	6	995	1	794	6	423
7	864	2	072	7	83059	2	854	7	480
8	938	3	140	8	123	3	914	8	536
9	77012	4	209	9	187	4	974	9	593
590	77085	635	80277	680	83251	725	86034	770	88649
1	159	6	346	1	315	6	094	1	705
2	232	7	414	2	378	7	153	3	762
3	305	8	482	3	442	8	213	3	818
4	379	9	550	4	506	9	273	4	874

No.	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Log.
775	88930	820	91381	865	93702	910	95904	955	98000
6	986	1	434	6	752	1	952	6	046
7	89042	2	487	7	802	2	999	7	091
8	098	3	540	8	852	3	96047	8	137
9	154	4	593	9	902	4	095	9	182
780	89209	825	91645	870	93952	915	96142	960	98227
1	265	6	698	1	94002	6	190	1	272
2	321	7	751	2	052	7	237	2	318
3	376	8	803	3	101	8	284	3	363
4	432	9	855	4	151	9	332	4	408
785	89487	830	91908	875	94201	920	96379	965	98453
6	542	1	960	6	250	1	426	6	498
7	597	2	92012	7	300	2	473	7	543
8	653	3	065	8	349	3	520	8	588
9	708	4	117	9	399	4	567	9	632
790	89763	835	92169	880	94448	925	96614	970	98677
1	818	6	221	1	498	6	661	1	722
2	873	7	273	2	547	7	708	2	767
3	927	8	324	3	596	8	754	3	811
4	982	9	376	4	645	9	802	4	856
795	90037	840	92428	885	94694	930	96848	975	98900
6	091	1	480	6	743	1	895	6	945
7	146	2	531	7	792	2	941	7	989
8	200	3	583	8	841	3	988	8	99034
9	255	4	634	9	890	4	97035	9	078
800	90309	845	92686	890	94939	935	97081	980	99123
1	363	6	737	1	988	6	128	1	167
2	417	7	788	2	95036	7	174	2	211
3	472	8	840	3	085	8	220	3	255
4	526	9	891	4	134	9	267	4	300
805	90580	850	92942	895	95182	940	97313	985	99344
6	634	1	993	6	231	1	359	6	388
7	687	2	93044	7	279	2	405	7	432
8	741	3	095	8	328	3	451	8	476
9	795	4	146	9	376	5	497	9	520
810	90849	855	93197	900	95424	945	97543	990	99564
1	902	6	247	1	472	6	589	1	607
2	956	7	298	2	521	7	635	2	651
3	91009	8	349	3	569	8	681	3	695
4	062	9	399	4	617	9	727	4	739
815	91116	860	93450	905	95665	950	97772	995	99782
6	169	1	500	6	713	1	818	6	826
7	222	2	551	7	761	3	864	7	870
8	275	3	601	8	809	3	909	8	913
9	328	4	651	9	856	4	955	9	957

TRIGONOMETRIC FUNCTIONS.

Angle.	Sines.		Cosines.		Tangents.		Cotangents.	
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.
0°	00000	− ∞	1.0000	10.00000	00000	− ∞	∞	+ ∞
10′	291	7.46373	1.0000	10.00000	291	7.46373	343.77	2.53627
20	582	7.76475	99998	9.99999	582	7.76476	171.89	2.23524
30	873	7.94084	996	998	873	7.94086	114.59	2.05914
40	01164	8.06578	993	997	01164	8.06581	85.940	1.93419
50	454	8.16268	989	995	455	8.16273	68.750	1.83727
1°	01745	8.24186	99985	9.99993	01746	8.24192	57.290	1.75808
10′	02036	8.30879	979	991	02036	8.30888	49.104	1.69112
20	327	8.36678	973	988	328	8.36689	42.964	1.63311
30	618	8.41792	966	985	619	8.41807	38.188	[illegible]58193
40	908	8.46366	958	982	910	8.46385	34.368	[illegible].53615
50	03199	8.50504	949	978	03201	8.50527	31.242	1.49473
2°	03490	8.54282	99939	9.99974	03492	8.54308	28.636	1.45692
10′	781	8.57757	929	969	783	8.57788	26.432	1.42212
20	04071	8.60973	917	964	04075	8.61009	24.542	1.38991
30	362	8.63968	905	959	366	8.64009	22.904	1.35991
40	653	8.66769	892	953	658	8.66816	21.470	1.33184
50	943	8.69400	878	947	949	8.69453	20.206	1.30547
3°	05234	8 71880	99863	9.99940	05241	8.71940	19.081	1.28060
10′	524	8.74226	847	934	533	8.74292	18.075	1.25708
20	814	8.76451	831	926	824	8.76525	17.169	1.23475
30	06105	8.78568	813	919	06116	8.78649	16.350	1 21351
40	395	8.80585	795	911	408	8.80674	15.605	1.19326
50	685	8.82513	776	903	700	8.82610	14.924	1.17390
4°	06976	8.84358	99756	9.99894	06993	8.84464	14.301	1.15536
10′	07266	8.86128	736	885	07285	8.86243	13.727	1.13757
20	556	8.87829	714	876	578	8.87953	13.197	1.12047
30	846	8.89464	692	866	870	8.89598	12.706	1.10402
40	08136	8.91040	668	856	08163	8.91185	12.251	1.08815
50	426	8.92561	644	845	456	8.92716	11.826	1.07284
5°	08716	8.94030	99619	9.99834	08749	8.94195	11.430	1.05805
10′	09005	8.95450	594	823	09042	8.95627	11.059	1.04373
20	295	8.96825	567	812	335	8.97013	10.712	1.02987
30	585	8.98157	540	800	629	8.98358	10.385	1.01642
40	874	8.99450	511	787	923	8.99662	10.078	1.00338
50	10164	9.00704	482	775	10216	9.00930	9.7882	0.99070
6°	10453	9.01923	99452	9.99761	10510	9.02162	9.5144	0.97838
10′	742	9.03109	421	748	805	9.03361	9.2553	0.96639
20	11031	9.04262	390	734	11099	9.04528	9.0098	0.95472
30	320	9.05386	357	720	394	9.05666	8.7769	0.94334
40	609	9.06481	324	705	688	9.06775	8.5555	0.93225
50	898	9.07548	290	690	983	9.07858	8.3450	0.92142

Angle.	Sines.		Cosines.		Tangents.		Cotangents.	
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.
7°	12187	9.08589	99255	9.99675	12278	9.08914	8.1443	0.91086
10′	476	9.09606	219	659	574	9.09947	7.9530	0.90053
20	764	9.10599	182	643	869	9.10956	7.7704	0.89044
30	13053	9.11570	144	627	13165	9.11943	7.5958	0.88057
40	341	9.12519	106	610	461	9 12909	7.4287	0.87091
50	629	9.13447	067	593	758	9.13854	7.2687	0.86146
8°	13917	9.14356	99027	9.99575	14054	9.14780	7.1154	0.85220
10′	14205	9.15245	98986	557	351	9.15688	6.9682	0.84312
20	493	9.16116	944	539	648	9.16577	6.8269	0.83423
30	781	970	902	520	945	9.17450	6.6912	0.82550
40	15069	9.17807	858	501	15243	9.18306	6.5606	0.81694
50	356	9.18628	814	482	540	9.19146	6.4348	0 80854
9°	15643	9.19433	98769	9.99462	15838	9.19971	6.3138	0.80029
10′	931	9.20223	723	442	16137	9.20782	6.1970	0.79218
20	16218	999	676	421	435	9.21578	6.0844	0.78422
30	505	9.21761	629	401	734	9.22361	5.9758	0.77639
40	792	9.22509	580	379	17033	9.23130	5.8708	0.76870
50	17078	9.23244	531	357	333	887	5.7694	113
10°	17365	9.23967	98481	9.99335	17633	9.24632	5.6713	0.75368
10′	651	9.24677	430	313	933	9.25365	5.5764	0.74635
20	937	9.25376	378	290	18233	9.26086	5.4845	0.73914
30	18224	9.26063	325	267	534	797	5.3955	203
40	509	739	272	243	835	9.27496	5.3093	0.72504
50	795	9.27405	218	219	19136	9.28186	5.2257	0.71814
11°	19081	9.28060	98163	9.99195	19438	9.28865	5.1446	0.71135
10′	366	705	107	170	740	9.29535	5.0558	0.70465
20	652	9.29340	050	145	20042	9.30195	4.9894	0.69805
30	937	966	97992	120	345	846	4.9152	154
40	20222	9.30582	934	093	648	9.31489	4.8430	0.68511
50	507	9.31189	875	067	952	9.32122	4.7729	0.67878
12°	20791	9.31788	97815	9.99040	21256	9.32747	4.7046	0 67253
10′	21076	9.32378	754	013	560	9.33365	4.6382	0.66635
20	360	960	692	9.98986	864	974	4.5736	026
30	644	9.33534	630	958	22169	9.34576	4.5107	0.65424
40	928	9.34100	566	930	475	9.35170	4 4494	0.64830
50	212	658	502	901	781	757	4.3897	243
13°	22495	9.35209	97437	9.98872	23087	9.36336	4.3315	0.63664
10′	778	752	371	843	393	909	4.2747	091
20	23062	9.36289	304	813	700	9.37476	4.2193	0.62524
30	345	819	237	783	24008	9.38035	4.1653	0.61965
40	627	9,37341	169	753	316	589	4.1126	411
50	910	858	100	722	624	9.39136	4.0611	0.60864

Angle.	Sines.		Cosines.		Tangents.		Cotangents.	
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.
14°	24192	9.38368	97030	9.98690	24933	9.39677	4.0108	0.60323
10′	474	871	96959	659	25242	9.40212	3.9617	0.59788
20	756	9.39369	887	627	552	742	3.9136	258
30	25038	860	815	594	862	9.41266	3.8667	0.58734
40	320	9.40346	742	561	26172	784	3.8208	216
50	601	825	667	528	483	9.42297	3.7760	0.57703
15°	25882	9.41300	96593	9.98494	26795	9.42805	3.7321	0.57195
10′	26163	768	517	460	27107	9.43308	3.6891	0.56692
20	443	9.42232	440	426	419	806	3.6470	194
30	724	690	363	391	732	9.44299	3.6059	0.55701
40	27004	9.43143	285	356	28046	787	3.5656	213
50	284	591	206	320	360	9.45271	3.5261	0.54729
16°	27564	9.44034	96126	9.98284	28675	9.45750	3.4874	0.54250
10′	843	472	046	248	990	9.46224	3.4495	0.53776
20	28123	905	95964	211	29305	694	3.4124	306
30	402	9.45334	882	174	621	9.47160	3.3759	0.52840
40	680	758	799	136	938	622	3.3402	378
50	959	9.46178	715	098	30255	9.48080	3.3052	0.51920
17°	29237	9.46594	95630	9.98060	30573	9.48534	3.2709	0.51466
10′	515	9.47005	545	021	891	984	3.2371	016
20	793	411	459	9.97982	31210	9.49430	3.2041	0.50570
30	30071	814	372	942	530	872	3.1716	128
40	348	9.48213	284	902	850	9.50311	3.1397	0.49689
50	625	607	195	861	32171	746	3.1084	254
18°	30902	9.48998	95106	9.97821	32492	9.51178	3.0777	0.48822
10′	31178	9.49385	015	779	814	606	3.0475	394
20	454	768	94924	738	33136	9.52031	3.0178	0.47969
30	730	9.50148	832	696	460	452	2.9887	548
40	32006	523	740	653	783	870	2.9600	130
50	282	896	646	610	34108	9.53285	2.9319	0.46715
19°	32557	9.51264	94552	9.97567	34433	9.53697	2.9042	0.46303
10′	832	629	457	523	758	9.54106	2.8770	0.45894
20	33106	991	361	479	35085	512	2.8502	488
30	381	9.52350	264	435	412	915	2.8239	085
40	655	705	167	390	740	9.55315	2.7980	0.44685
50	929	9.53056	068	344	36068	712	2.7725	288
20°	34202	9.53405	93969	9.97299	36397	9.56107	2.7475	0.43893
10′	475	751	869	252	727	498	2.7228	502
20	748	9.54093	769	206	37057	887	2.6985	113
30	35021	433	667	159	388	9.57274	2.6746	0.42726
40	293	769	565	111	720	658	2.6511	342
50	565	9.55102	462	063	38053	9.58039	2.6279	0.41961

Angle.	Sines.		Cosines.		Tangents.		Cotangents.	
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.
21°	35837	9.55433	93358	9.97015	38386	9.58418	2.6051	0.41582
10′	36108	761	253	9.96966	721	794	2.5826	206
20	379	9.56085	148	917	39055	9.59168	2.5605	0.40832
30	650	407	042	868	391	540	2.5386	460
40	921	727	92935	818	727	909	2.5172	091
50	37191	9.57044	827	767	40065	9.60276	2.4960	0.39724
22°	37461	9.57358	92718	9.96717	40403	9.60641	2.4751	0.39359
10′	730	669	609	665	741	9.61004	2.4545	0.38996
20	37999	978	499	614	41081	364	2.4342	636
30	38268	9.58284	388	562	421	722	2.4142	278
40	537	588	276	509	763	9.62079	2.3945	0.37921
50	805	889	164	456	42105	433	2.3750	567
23°	39073	9.59188	92050	9.96403	42447	9.62785	2.3559	0.37215
10′	341	484	91936	349	791	9.63135	2.3369	0.36865
20	608	778	822	294	43136	484	2.3183	516
30	875	9.60070	706	240	481	830	2.2998	170
40	40141	359	590	185	8.8	9.64175	2.2817	0.35825
50	408	646	472	129	44175	517	2.2637	483
24°	40674	9.60931	91355	9.96073	44523	9.64858	2.2460	0.35142
10′	939	9.61214	236	017	872	9.65197	2.2286	0.34803
20	41204	494	116	9.95960	45222	535	2.2113	465
30	469	773	90996	902	573	870	2.1943	130
40	734	9.62049	875	844	924	9.66204	2.1775	0.33796
50	998	323	753	786	46277	537	2.1609	463
25°	42262	9.62595	90631	9.95728	46631	9.66867	2.1445	0.33133
10′	525	865	507	668	985	9.67196	2.1283	0.32804
20	42788	9.63133	383	609	47341	524	2.1123	476
30	43051	398	259	549	698	850	2.0965	150
40	313	662	133	488	48055	9.68174	2.0809	0.31826
50	575	924	90007	427	414	497	2.0655	503
26°	43837	9.64184	89879	9.95366	48773	9.68818	2.0503	0.31182
10′	44098	442	752	304	49134	9.69138	2.0353	0.30862
20	359	698	623	242	495	457	2.0204	543
30	620	953	493	179	858	774	2.0057	226
40	880	9.65205	363	116	50222	9.70089	1.9912	0.29911
50	45140	456	232	052	587	404	1.9768	596
27°	45399	9.65705	89101	9.94988	50593	9.70717	1.9626	0.29283
10′	658	952	88968	923	51319	9.71028	1.9486	0.28972
20	917	9.66197	835	858	688	339	1.9347	661
30	46175	441	701	793	52057	648	1.9210	352
40	433	682	566	727	427	955	1.9074	045
50	690	922	431	660	798	9.72262	1.8940	0.27738

Angle.	Sines.		Cosines.		Tangents.		Cotangents.	
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.
28°	46947	9.67161	88295	9.94593	53171	9.72567	1.8807	0.27433
10′	47204	398	158	526	545	872	1.8676	128
20	460	633	020	458	920	9.73175	1.8546	0.26825
30	716	866	87882	390	54296	476	1.8418	524
40	971	9.68098	743	321	673	777	1.8291	223
50	48226	328	603	252	55051	9.74077	1.8165	0.25923
29°	48481	9.68557	87462	9.94182	55431	9.74375	1.8040	0.25625
10′	735	784	321	112	812	673	1.7917	327
20	989	9.69010	178	041	56194	969	1.7796	031
30	49242	234	036	9.93970	577	9.75264	1.7675	0.24736
40	495	456	86892	898	962	558	1.7556	442
50	748	677	748	826	57348	852	1.7447	148
30°	50000	9.69897	86603	9.93753	57735	9.76144	1.7321	0.23856
10′	252	9.70115	457	680	58124	435	1.7205	565
20	503	332	310	606	513	725	1.7090	274
30	754	547	163	532	905	9.77015	1.6977	0.22985
40	51004	761	015	457	59297	303	1.6864	697
50	254	973	85866	382	691	591	1.6753	409
31°	51504	9.71184	85717	9.93307	60086	9.77877	1 6643	0.22123
10′	753	393	567	230	483	9.78163	1.6534	0.21837
20	52002	602	416	154	881	448	1.6426	552
30	250	809	264	077	61280	732	1.6319	268
40	498	9.72014	112	9.92999	681	9.79015	1.6212	0.20985
50	745	218	84959	921	62083	297	1.6107	703
32°	52992	9.72421	84805	9.92842	62487	9.79579	1.6003	0.20421
10′	53238	623	650	763	892	860	1.5900	140
20	484	823	495	683	63299	9.80140	1.5798	0.19860
30	730	9.73022	339	603	707	419	1.5697	581
40	975	219	182	522	64117	697	1.5597	303
50	52220	416	025	441	528	975	1.5497	025
33°	54464	9.73611	83867	9.92359	64941	9.81252	1.5399	0.18748
10′	708	805	708	277	65355	528	1.5301	472
20	951	997	549	194	771	803	1.5204	197
30	55194	9.74189	389	111	66189	9.82078	1,5108	0.17922
40	436	379	228	027	608	352	1.5013	648
50	678	568	066	9.91942	67028	626	1.4919	374
34°	55919	9.74756	82904	9.91857	67451	9.82899	1.4826	0.17101
10′	56160	943	741	772	875	9.83171	1.4733	0.16829
20	401	9.75128	577	686	68301	442	1.4641	558
30	641	313	413	599	728	713	1.4550	287
40	880	496	248	512	69157	984	1.4460	016
50	57119	678	082	425	588	9.84254	1.4370	0.15746

Angle.	Sines.		Cosines.		Tangents.		Cotangents.	
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.
35°	57358	9.75859	81915	9.91336	70021	9.84523	1.4281	0.15477
10′	596	9.76039	748	248	455	791	1.4193	209
20	833	218	580	158	891	9.85059	1.4106	0.14941
30	58070	395	412	069	71329	327	1.4019	673
40	307	572	242	9.90978	769	594	1.3934	406
50	543	747	072	887	72211	860	1.3848	140
36°	58779	9.76922	80902	9.90796	72654	9.86126	1.3764	0.13874
10′	59014	9.77095	730	704	73100	392	1.3680	608
20	248	268	558	611	547	656	1.3597	344
30	482	439	386	518	996	921	1.3514	079
40	716	609	212	424	74447	9.87185	1.3432	0.12815
50	949	778	038	330	900	448	1.3351	0.12552
37°	60182	9.77946	79864	9.90235	75355	9.87711	1.3270	0.12289
10′	414	9.78113	688	139	812	974	1.3190	026
20	645	280	512	043	76272	9.88236	1.3111	0.11764
30	876	445	335	9.89947	733	498	1.3032	502
40	61107	609	158	849	77196	759	1.2954	241
50	337	772	78980	752	661	9.89020	1.2876	0.10980
38°	61566	9.78934	78801	9.89653	78129	9.89281	1.2799	0.10719
10′	795	9.79095	622	554	598	541	1.2723	459
20	62024	256	442	455	79070	801	1.2647	199
30	251	415	261	354	544	9.90061	1.2572	0.09939
40	479	573	079	254	80020	320	1.2497	680
50	706	731	77897	152	498	578	1.2423	422
39°	62932	9.79887	77715	9.89050	80978	9.90837	1.2349	0.09163
10′	63158	9.80043	531	9.88948	81461	9.91095	1.2276	0.08905
20	383	197	347	844	946	353	1.2203	647
30	608	351	162	741	82434	610	1.2131	390
40	832	504	76977	636	923	868	1.2059	132
50	64056	656	791	531	83415	9.92125	1.1988	0.07875
40°	64279	9.80807	76604	9.88425	83910	9.92381	1.1918	0.07619
10′	501	957	417	319	84407	638	1.1847	362
20	723	9.81106	229	212	906	894	1.1778	106
30	945	254	041	105	85408	9.93150	1.1708	0.06850
40	65166	402	75851	9.87996	912	406	1.1640	594
50	386	549	661	887	86419	661	1.1571	339
41°	65606	9.81694	75471	9.87778	86929	9.93916	1.1504	0.06084
10′	825	839	280	668	87441	9.94171	1.1436	0.05829
20	66044	983	088	557	955	426	1.1369	574
30	262	9.82126	74896	446	88473	681	1.1303	319
40	480	269	703	334	992	935	1.1237	065
50	697	410	509	221	89515	9.95190	1.1171	0.04810

Angle.	Sines.		Cosines.		Tangents.		Cotangents.	
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.
42°	66913	9.82551	74314	9.87107	90040	9.95444	1.1106	0.04556
10′	67129	691	120	9.86993	569	698	1.1041	302
20	344	830	73924	879	91099	952	1.0977	0.04048
30	559	968	728	763	633	9.96205	1.0913	0.03795
40	773	9.83106	531	647	92170	459	1.0850	541
50	987	242	333	530	709	712	1.0786	288
43°	68200	9.83378	73135	9.86413	93252	9.96966	1.0724	0.03034
10′	412	513	72937	295	797	9.97219	1.0661	0.02781
20	624	648	737	176	94345	472	1.0599	528
30	835	781	537	056	896	725	1.0538	275
40	69046	914	337	9.85936	95451	978	1.0477	022
50	256	9.84046	136	815	96008	9.98231	1.0416	0.01769
44°	69466	9.84177	71934	9.85693	96569	9.98484	1.0355	0.01516
10′	675	308	732	571	97133	737	1.0295	263
20	883	437	529	448	700	989	1.0235	011
30	70031	566	325	324	98270	9.99242	1.0176	0.00758
40	298	694	121	200	843	495	1.0117	505
50	505	822	70916	074	99420	747	1.0058	253
45°	70711	9.84948	70711	9.84948	1.0000	10.00000	1.0000	0.00000

NUMBERS OFTEN USED IN CALCULATIONS.

				LOGARITH.
Ratio of the circum. of a circle to its diameter.		$=\pi=$	3.1415926536	0.4971499
Vol. of sphere to radius	1 =	$\frac{4}{3}\pi=$	4.1887902048	0.6220886
Diameter of circle to area	1 =	$\sqrt{4\div\pi}=$	1.1283791671	0.0524550
Diam. of sphere to volume	1 =	$\sqrt[3]{6\div\pi}=$	1.2407009818	0.0936671
		$\sqrt{\pi}=$	1.7724538509	0.2485749
		$\pi^2=$	9.8966044011	0.9942997
Base of Naperian log......	=	$e=$	2.7182818285	0.4342945
Modulus of common logs..	=	$M=$	.4342944819	$\bar{1}.6377843$
Unit of circular measure..	=	$\frac{180°}{\pi}=$	57°.2957795130	1.7581226
Sin 1″		=	.0000048481	$\bar{6}.6855716$
Mean diam. of the earth..		=	7912 miles	3.8982863

ANSWERS AND RESULTS.

EXERCISE I. (Page 13.)

1. AB and BD, AC and CD, AE and ED, etc.

2. AB, AC, AE; DB and DC each negative, DE positive.

7. $\sqrt{13}$, $\sqrt{13}$, $\sqrt{13}$, $\sqrt{13}$, $\sqrt{34}$, 4, 3, 0.

11. (1) 1st and 4th. (2) 1st and 2nd. (3) 1st and 3rd.

EXERCISE II. (Page 21.)

3. $(4n + \frac{1}{3})$ right angles. 4. .2587469, 1.413966, .371483.

5. $6^\circ\ 51'\ 45''$, $111^\circ\ 6'\ 19''.944$, $271^\circ\ 7'\ 30''$, $4'\ 13''$.
$7^g\ 62^{\backprime}\ 50^{\backprime\backprime}$, $123^g\ 45^{\backprime}\ 6^{\backprime\backprime}$, $301^g\ 25^{\backprime}$, $7^{\backprime}\ 50^{\backprime\backprime}$.

6. $83^\circ\ 8'\ 15''$, $173^\circ\ 8'\ 15''$; $-(21^\circ\ 6'\ 19''.944)$, $68^\circ\ 53'\ 40''.056$; $-(181^\circ\ 7'\ 30'')$, $-(91^\circ\ 7'\ 30'')$; $89^\circ\ 55'\ 47''$, $179^\circ\ 55'\ 47''$.

7. 60°, 105°. 8. 60°, $133\frac{1}{3}^g$. 9. 16. 10. $4\frac{1}{2}^\circ$.

12. 9°, 90^g. 13. 25°, 27°; 65°, 63°.

14. $14\frac{6}{11}$ minutes after 11, or 20 minutes before 12.

15. 6°. 16. $15'$. 17. $30'\ 56''.734$. 18. 1170°, 6060°.

19. Sept. 2nd, P.M. $11^h\ 13^m\ 13.97^s$. 20. 126°, 422.4.

21. $37\frac{1}{2}^\circ$. 22. $2\frac{1}{7}$ inches. 23. 45°, 60°, 75°.

24. $A = 75^\circ$, $B = 81^\circ$, $C = 105^\circ$, $D = 99^\circ$. 27. $A = 60^\circ$, $B = 80^\circ$, $C = 40^\circ$.

EXERCISE III. (Page 31.)

1. (1) $\frac{\pi}{6}$. (2) $\frac{\pi}{4}$. (3) $\frac{\pi}{144}$. (4) $\frac{\pi^2}{180}$. (5) 1.
(6) $\frac{203\pi}{64,800,000}$. (7) $\frac{758325\pi}{2,000,000}$. (8) $\frac{1}{2}$.

2. (1) $180°$. (2) $60°$. (3) $180°$. (4) $\frac{180\theta}{\pi}$. (5) 20π.
(6) $\frac{180°}{\pi^2}$. (7) $117° \, 1' \, 36''$. (8) $101° \, 33' \, 14''.8$.

3. (1) 2.618 ft. (2) 13.09 ft. (3) .003515 ft. (4) 9.92103.

4. (1) $\frac{15}{\pi} = 4.7746$. (2) 572.957 ft. (3) 2.715 ft. (4) 10 ft.

5. (1) $3°.1831$. (2) $8°.594$. (3) $343°.774$. (4) $36°$.

6. $229°.183$, $254^g.648$, 4^r. 8. 6 hours.

9. $\frac{3\pi}{4}$, $\frac{5\pi}{6}$, $\frac{9\pi}{10}$, $\frac{(n-2)\pi}{n}$. 10. $47°.0281$. 11. $\frac{7\pi}{36}$. 12. $\frac{1}{3}$.

14. $30°$, $60°$, $90°$. 15. 16, 24. 17. 82.873 miles.

18. 400:1. 19. 473:489. 20. $\frac{1800\pi}{19\pi + 1800}$.

21. $\frac{\pi a}{a+b+c}$, $\frac{\pi b}{a+b+c}$, $\frac{\pi c}{a+b+c}$. 22. $\frac{90}{\pi^2} = 9°.118$.

23. $\frac{\pi}{2}$. 24. $\frac{\pi}{6}$. 25. $\frac{\pi bc}{bc+ca+ab}$, etc.

26. $72°$. 27. $\frac{2\pi a}{3\sqrt{3}}$.

EXERCISE IV. (Page 40.)

1. $\sin A = \cos B = \frac{3}{5}$, $\cos A = \sin B = \frac{4}{5}$, $\tan A = \cot B = \frac{3}{4}$.
$\cot A = \tan B = \frac{4}{3}$, $\sec A = \text{cosec } B = \frac{5}{4}$, $\text{cosec } A = \sec B = \frac{5}{3}$.

3. $\frac{2}{5}$, $-\frac{4}{3}$, $\frac{4}{3}$, $\frac{2}{5}$. 4. (1) $\frac{5}{13}$, $\frac{12}{13}$. (2) $\frac{5}{13}$, $-\frac{12}{13}$ or $-\frac{5}{13}$, $\frac{12}{13}$.

5. (1) $\frac{1}{\sqrt{2}}$. (2) 1. (3) $\sqrt{2}$. (4) $-\frac{1}{\sqrt{2}}$. (5) 1. (6) -1.
(7) $\frac{1}{\sqrt{2}}$. (8) $\frac{1}{\sqrt{2}}$. 6. $\frac{\sqrt{3}}{2}$, $60°$, $30°$. 7. 2, 337.5.

8. (1) $\frac{12}{37}$, (2) $\frac{12}{13}$, (3) $\frac{5}{13}$, (4) $-\frac{5}{13}$, (5) $\frac{5}{13}$, (6) $\frac{35}{37}$,
(7) $\frac{360}{481}$, (8) $\frac{360}{319}$, (9) $\frac{-12}{5}$, (10) $\frac{5}{12}$, (11) $-\frac{13}{5}$, (12) $\frac{13}{12}$.

9. See ans. to Ex. 1 of this exercise. 10. See Art. 50.

EXERCISE V. (PAGE 42.)

	sine.	cosine.	tan.	cot.	sec.	cosec.
1. (1)	$\frac{5}{13}$,	$\frac{12}{13}$,	$\frac{5}{12}$,	$\frac{12}{5}$,	$\frac{13}{12}$,	$\frac{13}{5}$.
(2)	$\frac{7}{25}$,	$\frac{24}{25}$,	$\frac{7}{24}$,	$\frac{24}{7}$,	$\frac{25}{24}$,	$\frac{25}{7}$.
(3)	$\frac{2\sqrt{2}}{3}$,	$\frac{1}{3}$,	$2\sqrt{2}$,	$\frac{1}{2\sqrt{2}}$,	3,	$\frac{3}{2\sqrt{2}}$.

(4) $\frac{m^2-n^2}{m^2+n^2}$, $\frac{2mn}{m^2+n^2}$, $\frac{m^2-n^2}{2mn}$, etc.

(5) $\frac{\sqrt{6}\pm\sqrt{2}}{4}$, $\frac{\sqrt{6}\mp\sqrt{2}}{4}$, $2\pm\sqrt{3}$, $2\mp\sqrt{3}$, $\sqrt{6}\pm\sqrt{2}$, $\sqrt{6}\mp\sqrt{2}$.

(6) $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, $\sqrt{3}$, $\frac{1}{\sqrt{3}}$, 2, $\frac{2}{\sqrt{3}}$.

2. $-\frac{4}{5}$, $-\frac{3}{4}$, $\frac{5}{3}$. 3. $\frac{99}{101}$, $-\frac{20}{101}$. 5. $\frac{a}{c}$, $\frac{\sqrt{c^2-a^2}}{a}$.

6. $\sqrt{m^2+1}$, $\frac{\sqrt{m^2+1}}{m}$. 7. $\frac{x}{\sqrt{x^2+1}}$, $\frac{1}{\sqrt{x^2+1}}$. 8. $4\frac{1}{2}$, $3\frac{3}{4}$.

EXERCISE VI. (PAGE 46.)

32. $2\pm\sqrt{3}$, $\sqrt{6}\pm\sqrt{2}$. 33. $\pm\frac{1}{\sqrt{2}}$, ± 1. 34. $\pm\frac{1}{2}$, $\sqrt{3}$.

35. $\frac{\sqrt{5}-1}{4}$, $\sqrt{5+2\sqrt{5}}$. 36. $\frac{25}{12}$ or $\frac{13}{6}$. 37. $\sqrt{a^2+b^2}$.

38. a, or b.

39. $\tan^2 x = \cos^2 a \tan^2 b - \sin^2 a$.

$$\tan^2 y = \frac{\sin^2 a \sec^2 b}{\cos^2 a \tan^2 b - \sin^2 a}.$$

40. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

41. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. 42. $(1-x^2)(1+y^2)=1$.

43. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$. 44. $x^{\frac{2}{3}}y^{\frac{2}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}})=1$.

45. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=(xy)^{\frac{4}{3}}$. 46. $x^2+y^2=a^2+b^2$.

47. $(ab-1)^2+(a+1)^2=(b+1)^2$.

EXERCISE VII. (PAGE 50.)

2. (1) $\cos\theta\sec\theta = 1$. (2) $\tan\theta\cot\theta = 1$.
(3) $\cos\theta\tan\theta = \sin\theta$. (4) $\sin\theta\cot\theta = \cos\theta$.

3. $PN.AT.Ot = OP^3$, $\sin\theta\cot\theta\sec\theta = 1$, $\cos\theta\tan\theta\operatorname{cosec}\theta = 1$.

4. (1) $\frac{5\pi\sqrt{2}}{4}$. (2) $\frac{15\pi\sqrt{2}}{4}$ (3) $\frac{25\pi\sqrt{2}}{4}$. (4) $\frac{35\pi\sqrt{2}}{4}$.

5. $\frac{60}{13}$, $\frac{25}{13}$, $\frac{25}{12}$; 5π.

EXERCISE VIII. (PAGE 59.)

	angle.	sin.	tan.	sec.	cos.	cot.	cosec.
1.	$135°$	$\frac{1}{\sqrt{2}}$,	-1,	$-\sqrt{2}$,	$-\frac{1}{\sqrt{2}}$,	-1,	$\sqrt{2}$.
	$-45°$	$-\frac{1}{\sqrt{2}}$,	-1,	$\sqrt{2}$,	$\frac{1}{\sqrt{2}}$,	-1,	$-\sqrt{2}$.
2.	$120°$	$\frac{\sqrt{3}}{2}$,	$-\sqrt{3}$,	-2,	$-\frac{1}{2}$,	$-\frac{1}{\sqrt{3}}$,	$\frac{2}{\sqrt{3}}$.
	$150°$	$\frac{1}{2}$,	$-\frac{1}{\sqrt{3}}$,	$-\frac{2}{\sqrt{3}}$,	$-\frac{\sqrt{3}}{2}$,	$-\sqrt{3}$,	2.
	$210°$	$-\frac{1}{2}$,	$\frac{1}{\sqrt{3}}$,	$-\frac{2}{\sqrt{3}}$,	$-\frac{\sqrt{3}}{2}$,	$\sqrt{3}$,	-2.
	$-30°$	$-\frac{1}{2}$,	$-\frac{1}{\sqrt{3}}$,	$\frac{2}{\sqrt{3}}$,	$\frac{\sqrt{3}}{2}$,	$-\sqrt{3}$,	-2.

	angle.	sin.	tan.	sec.
3.	$22\frac{1}{2}°$	$\frac{1}{2}\sqrt{2-\sqrt{2}}$,	$\sqrt{2}-1$,	$\sqrt{4-2\sqrt{2}}$,
	$67\frac{1}{2}°$	$\frac{1}{2}\sqrt{2+\sqrt{2}}$,	$\sqrt{2}+1$,	$\sqrt{4+2\sqrt{2}}$,

cos.	cot.	cosec.
$\frac{1}{2}\sqrt{2+\sqrt{2}}$,	$\sqrt{2}+1$,	$\sqrt{4+2\sqrt{2}}$.
$\frac{1}{2}\sqrt{2-\sqrt{2}}$,	$\sqrt{2}-1$,	$\sqrt{4-}$ [illegible]

	angle.	sin.	tan.	sec.	cos.	cot.	co[illegible]
4.	$270°$	-1,	∞,	∞,	0,	0,	-1

5. $45°$, $135°$, $225°$, $315°$. 6. $\frac{\sqrt{3}}{2}$. 7. 1.

11. (1) $0°, 60°$. (2) $\pm 30°, \pm 150°$. (3) $30°, 60°$. (4) $90°$.
(5) $90°, 30°$. (6) $18°, -54°$. (7) 60. (8) $\pm 30°, \pm 45°$.
(9) $135°, -45°$. (10) $22\frac{1}{2}°$. (11) $15°, 75°$.
(12) $18°, 162°$. (13) $30°, 45°, 150°$. (14) $30°, 60°$.
(15) $A = 45°, B = 15°$. (16) $A = 52\frac{1}{2}°, B = 7\frac{1}{2}°$.
(17) $A = 135°, B = 60°$. (18) $A = 30°, B = 15°, C = 45°$.

12. $(\sqrt{3} - \sqrt{2})(\sqrt{2} - 1), (\sqrt{3} - \sqrt{2})(\sqrt{2} + 1)$. 13. $45°$ and $60°$.

16. (1) Impossible, (2) Possible. 17. (1) Possible, (2) Impossible.

18. (1) $(a^2 b)^{\frac{2}{3}} + (ab^2)^{\frac{2}{3}} = 1$. (2) $\cancel{\dfrac{a-m}{a-b} + \dfrac{n-a}{a+b}} = 1$.

$\dfrac{am - bn + bm + an}{a^2 - b^2} = 1$

19. $\dfrac{(a-m)(a-n)}{(b-m)(b-n)} = \dfrac{a^2}{b^2}$.

EXERCISE IX. (Page 63.)

1. $a = 10\sqrt{3}, b = 10, A = 60°$.
2. $b = 15(2 + \sqrt{3}), c = 15(\sqrt{6} + \sqrt{2})$.
3. $b = \sqrt{10 - 4\sqrt{5}}, c = 2\sqrt{5 - \sqrt{5}}$.
4. $c = \sqrt{2 + \sqrt{2}}, a = \frac{1}{2}(2 + \sqrt{2}), p = \frac{1}{4}\sqrt{4 + 2\sqrt{2}}$.
5. $4\sqrt{3}$. 6. $50(\sqrt{2} + 1)$. 7. $50(\sqrt{6} - \sqrt{2})$.
8. $50(\sqrt{5} - 1), 2500\sqrt{5 - 2\sqrt{5}}$. 9. $\dfrac{2p^2}{\sqrt{3}}$.
10. $10(\sqrt{6} - \sqrt{2}), 10(\sqrt{6} + \sqrt{2})$.
11. $3(\sqrt{6} + \sqrt{2}), 3\sqrt{2}, \frac{9}{2}(3 - \sqrt{3})$.
12. 5 feet, 3 feet. 13. $6(\sqrt{3} + 2)$. 14. $30°$.
15. $\dfrac{1}{\cot 22\frac{1}{2}° - \cot 30°} = \dfrac{1 + \sqrt{2} + \sqrt{3}}{2\sqrt{2}}$ mls. 16. $10\sqrt{19 - 2\sqrt{5}}$.

17. Draw AD perpendicular to BC, then $b \sin C = AD$ and $c \cos B = BD$, from which the results of this and the following example easily follow.

19. 200. 21. $r\sqrt{2 - \sqrt{2}}, \dfrac{r}{2}\sqrt{2 + \sqrt{2}}, 2r^2\sqrt{2}$.

22. $\dfrac{a\sqrt{3}}{3b}, 45°$. 23. 170 feet.

24. $\dfrac{d}{(\cot^2\beta - \cot^2\alpha)^{\frac{1}{2}}}$, $\dfrac{d\cot\alpha}{\cot^2\beta - \cot^2\alpha}$. 25. $50\sqrt{2}$.

26. $90°$, $72°$. 30. $10\sqrt{8+2\sqrt{3}}$, $\frac{10}{3}(3\sqrt{2} - \sqrt{6})$.

EXERCISE X. (PAGE 70.)

24. $\dfrac{-b \pm \sqrt{a^2+b^2}}{a}$. 29. $2nr\sin\dfrac{\pi}{n}$, $2nr\tan\dfrac{\pi}{n}$.

30. $\frac{1}{2}nr^2\sin\dfrac{2\pi}{n}$, $nr^2\tan\dfrac{\pi}{n}$. 33. $b\sqrt{\dfrac{a+b}{a-b}}$. 34. $20\sqrt{3}$, $30\sqrt{3}$.

EXERCISE XI. (PAGE 77.)

1. .24624, $22°\ 16'\ 8''$.
2. .81672, $75°\ 38'\ 56''$.
3. .81740, $38°\ 16'\ 57''$.
4. 1.0634, $20°\ 28'\ 25''$.
5. .81698, $-.57667$.
6. 4.8053, $80°\ 53'\ 30''$.
7. $36°\ 52'\ 11''$, $53°\ 7'\ 49''$, $90°$.
8. $119°\ 14'\ 45''$.
9. -1.7860.
10. $112°\ 29'\ 13''$, .92397.
11. $A = 41°\ 36'\ 36''$, or $89°\ 47'\ 59''$.
    $B = \quad 12'\ 1''$, or $48°\ 23'\ 24''$.
12. $A = 71°\ 5'$.
    $B = 34°\ 35'$.

EXERCISE XII. (PAGE 86.)

1. $a = 10.353$, $b = 14.641$, $C = 105°$.
2. $a = 13$ feet, $\sin B = \dfrac{7\sqrt{3}}{26}$, $\sin C = \dfrac{15\sqrt{3}}{26}$.
3. $\frac{5}{7}$, $\dfrac{2\sqrt{6}}{5}$, $6\sqrt{6}$. 4. $\sin\dfrac{A}{2} = \dfrac{1}{\sqrt{6}}$, $\cos\dfrac{B}{2} = \dfrac{4}{\sqrt{21}}$, $\tan\dfrac{C}{2} = \dfrac{\sqrt{5}}{3}$.
5. 14, 84. 6. $55°\ 35'\ 2''$, $59°\ 47'\ 38''$, $64°\ 37'\ 20''$.
7. $18°\ 55'\ 28''$, 12. 8. 6.1926, $88°\ 38'\ 29''$, $53°\ 6'\ 31''$.
9. $161°\ 20'\ 55''$, $2°\ 9'\ 5''$. 10. $45°$, $60°$, $75°$. 11. $120°$.
12. .8; $30°$, or $120°$. 13. $B = 90°$, $C = 72°$, $c = 4\sqrt{5+2\sqrt{5}}$.

14. $B=45^\circ$ or 135°, $C=120^\circ$ or 30°, $c=2\,(3\sqrt{2}+\sqrt{6})$, or $2\,(\sqrt{2}+\sqrt{6})$.

15. $A=54^\circ$ or 126°, $B=108^\circ$ or 36°.

16. $\frac{1}{2}\sqrt{190}$. 17. $20\sqrt{3}$; $54^\circ\ 44'\ 10''$. 18. 17.76 rods.

26. $B=45^\circ$. 31. $A=60^\circ$; 28, 20, 12. 32. $\sqrt{abc\,(a+b+c)}$.

33. 2525, 2710. 34. 815.85. 35. $37^\circ\ 27'\ 12''$.

EXERCISE XIII. (PAGE 97.)

1. 2, 5, 4, 6, 12. 2. 1:2. 3. 30°.

4. 109.5568. 5. $(s-a)\sec\frac{A}{2}$, etc. 6. $a\sec\frac{A}{2}$, etc.

7. $a\operatorname{cosec}\frac{A}{2}$, etc. 8. $\frac{1}{2}(B+C)$, etc. 9. $\frac{abcs}{2S}=\frac{as}{\sin A}$.

EXERCISE XV. (PAGE 110.)

1. $(4n+1)\frac{\pi}{2}$. 2. $2n\pi$. 3. $n\pi+\frac{\pi}{4}$. 4. $n\pi+\frac{3\pi}{4}$.

5. $n\pi+\frac{\pi}{6}$. 6. $n\pi+(-1)^n\frac{\pi}{6}$. 7. $2n\pi\pm\frac{\pi}{6}$.

8. $n\pi\pm\frac{\pi}{6}$. 9. $n\pi\pm\frac{\pi}{4}$. 10. $n\pi+\frac{\pi}{4}$. 11. $n\pi\pm\frac{\pi}{4}$.

12. $n\pi\pm a$. 13. $\frac{1}{2}\left\{n\pi+(-1)^n\frac{\pi}{6}\right\}$.

14. $n\pi$ or $2n\pi\pm\frac{\pi}{3}$. 15. $\frac{(2n+1)\pi}{2(p+q)}$. 16. Impossible.

17. $(6n\pm1)\frac{\pi}{3}$ 18. $n\pi+(-1)^n\frac{\pi}{6}$.

19. $(2n+1)\frac{\pi}{2}$ or $\frac{1}{4}\left\{n\pi+(-1)^n\frac{\pi}{6}\right\}$. 20. $(4n+3)\frac{\pi}{4}$.

21. $\left(n+\frac{1}{2}\right)\frac{\pi}{2}$ or $\frac{1}{7}\left\{n\pi+(-1)^n\frac{\pi}{6}\right\}$.

22. $A=n\pi+\frac{1}{2}\left\{\frac{\pi}{4}+(-1)^n\frac{\pi}{3}\right\}$.

$B=\quad\frac{1}{2}\left\{(-1)^n\frac{\pi}{3}-\frac{\pi}{4}\right\}$.

25. $2n\pi+a$. 26. $(6n\pm1)\,\frac{15}{88}$ seconds.

EXERCISE XVI. (PAGE 117.)

2. 1, 0. 3. (1) $\frac{56}{65}$, $\frac{63}{65}$; $-\frac{16}{65}$, $-\frac{23}{63}$. 4. 315°. 7. 1.
8. $\frac{1}{239}$. 9. $\frac{120}{119}$. 14. $2m-1$.

EXERCISE XVIII. (PAGE 122.)

1. $2\sin 45^\circ \cos 30^\circ$.
2. $2\cos 45^\circ \sin 30^\circ$.
3. $2\cos 45^\circ \cos 15^\circ$.
4. $2\sin 45^\circ \sin 15^\circ$.
5. $2\sin 67\tfrac{1}{2}^\circ \cos 7\tfrac{1}{2}^\circ$.
6. $-2\sin 70^\circ \sin 10^\circ$.
7. $2\sin 2\theta \cos\theta$.
8. $-2\sin 2\theta \sin\theta$.
9. $2\cos\frac{\pi}{2}\cos\frac{\pi}{6}$.
10. $2\sin\frac{\pi}{4}\cos\frac{\pi}{12}$.
11. $\sin(\theta+\phi)+\sin(\theta-\phi)$.
12. $\cos(a-\beta)+\cos(a+\beta)$.
13. $\frac{1}{2}\{\cos(2\theta-2\phi)-\cos(2\theta+2\phi)\}$.
14. $\frac{1}{2}(\cos 2\theta+\cos 4\theta)$.
15. $\frac{1}{2}(\cos 4a-\cos 6a)$.
16. $\frac{1}{2}(\cos\theta-\cos 4\theta)$.
17. $\frac{1}{2}(\cos 2B-\cos 2A)$.
18. $\frac{1}{2}(\cos 2B+\cos 2A)$.
19. $-\frac{1}{4}(2\cos 2A+1)$.
20. $-\sin 3A$.
42. $(2n+1)\frac{\pi}{2}$, $\frac{n\pi}{2}$.
43. $\frac{n\pi}{3}$ or $\frac{1}{4}\left\{2n\pi\pm\frac{\pi}{3}\right\}$.
44. $\frac{n\pi}{2}$ or $2n\pi\pm\frac{2\pi}{3}$.
45. $n\pi$, $\frac{n\pi}{5}$, or $n\pi\pm\frac{\pi}{3}$.
46. $n\pi$, or $\frac{1}{3}\left\{n\pi+(-1)^n\frac{\pi}{6}\right\}$.
47. $n\pi\pm\frac{\pi}{6}$, or $(2n+1)\frac{\pi}{2}$.
48. $(10n\pm1)\frac{\pi}{5}$, or $(2n+1)\pi\pm\frac{\pi}{5}$.

EXERCISE XX. (PAGE 140).

1. (3) $\frac{1}{2}\sqrt{2-\sqrt{2}}$, $-\frac{1}{2}\sqrt{2+\sqrt{2}}$.

(4) $-\frac{1}{4}\sqrt{8-2\sqrt{10+2\sqrt{5}}}$, $-\frac{1}{4}\sqrt{8+2\sqrt{10+2\sqrt{5}}}$.

(5) $-\frac{1}{4}\sqrt{10-2\sqrt{5}}$, $-\frac{1}{4}(\sqrt{5}+1)$.

3. $\sin 9^\circ = \frac{1}{4}\{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}\}$, $\cos 9^\circ$
$= \frac{1}{4}\{\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}\}$.

$\tan 9^\circ = \operatorname{cosec} 18^\circ - \cot 18^\circ = \sqrt{5}+1 - \sqrt{5+2\sqrt{5}}$.

$\cot 9^\circ = \operatorname{cosec} 18^\circ + \cot 18^\circ = \sqrt{5}+1 + \sqrt{5+2\sqrt{5}}$.

$\sec 9^\circ = \sqrt{1+\tan^2 9^\circ} = \sqrt{7+3\sqrt{5}} - \sqrt{5+\sqrt{5}}$.

$\operatorname{cosec} 9^\circ = \sqrt{1+\cot^2 9^\circ} = \sqrt{7+3\sqrt{5}} + \sqrt{5+\sqrt{5}}$.

4. $\sqrt{6}-\sqrt{3}+\sqrt{2}-2$, $\sqrt{6}+\sqrt{3}-\sqrt{2}-2$.

5. $\sin A = \pm\frac{4}{5}$, $\cos A = \pm\frac{3}{5}$. 6. 45° and 135°.

7. 225° and 315°. 16. ± 1.

EXERCISE XXI. (PAGE 151.)

1. $\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, \frac{4}{3}$. 2. $\frac{\sqrt{3}+1}{2\sqrt{2}}$, 1. 4. $\frac{1}{3}, \frac{1}{5}$.

5. $\theta = 2\left(\frac{\pi}{2}-\theta\right)$, $\frac{\sqrt{3}}{2}, \frac{\pi}{3}$. 7. 225°, 405°, etc.

13. ∞; 1, or $2x^2-1$.

14. (1) -1, or $\frac{1}{6}$. (2) $\frac{\sqrt{6}-\sqrt{2}}{4}$, or $-\frac{1}{2}\sqrt{2-\sqrt{2}}$.

(3) 0, or $\pm\frac{1}{2}$. (4) ± 1, or $\pm(1\pm\sqrt{2})$.

(5) $x^2 = \frac{2}{17}(5-2\sqrt{2})$ (6) $x^2 = \frac{\sqrt{5}-1}{2}$.

(7) $\frac{a+b}{1-ab}$. (8) Impossible. (9) 0, or $\pm\frac{1}{2}$.

15. $A = \frac{1}{2}\sin^{-1}\frac{2a}{1+a^2}$. 16. $\frac{a+b}{1-ab}$. 19. $y = \frac{x}{1+x^2}$.

20. $\sqrt{2}$. 21. ± 1.

22. (1) $(n-\frac{1}{3})\pi + (-1)^n\frac{\pi}{6}$. (2) $\frac{\pi}{6} + n\pi + (-1)^n\frac{\pi}{4}$.

(3) $(2n\pm\frac{1}{3})\pi + 53^\circ\,7'\,47''$. (4) $n\pi$, or $2(n\pi - 69^\circ\,26'\,38'')$

(5) $(2n+1)\pi \pm 23^\circ\,25'\,43''$. (6) Impossible.

23. (1) $\sqrt{2} \sec \phi \cos \left(\phi - \frac{\pi}{4}\right)$, where $\tan \phi = \sin A$.

(2) $\sqrt{2} \sec \phi \cos \left(\phi - \frac{\pi}{4}\right)$, where $\tan \phi = a \sin A$.

(3) $a \sqrt{2} \sin A \cos \phi \sec \left(\phi - \frac{\pi}{4}\right)$, where $\tan \phi = a \cos A$.

25. Given expression $= \cot(\theta - 45^\circ)$.

28. $1{\cdot}85$.

29. a, or $a^2 - a + 1$.

30. $\pm ab$.

31. $\tan^{-1} \frac{1}{4} \left\{ -(2n+1)\pi \pm \sqrt{(2n+1)^2 \pi^2 + 16} \right\}$.

EXERCISE XXII. (PAGE 159.)

1. 2, 4, 8, 10, -3, -5.

2. 4, 5, -1, -2, $\frac{3}{2}$.

3. 3, 1, 0, -1, -3.

4. $-\frac{2}{3}$, $3\frac{3}{5}$.

5. $2n$, n, 2.

6. $2\frac{2}{3}$, $-\frac{7}{6}$, $\frac{7}{15}$, $60\frac{3}{4}$.

7. $2 \log a + 3 \log b + \log c$, $x \log a - y \log b + z \log c$, $-\left(\frac{7}{12} \log a + \log b\right)$.

8. $\log 5$.

9. $\frac{11}{60} \log 5 - \frac{3}{5} \log 2 - \frac{1}{10} \log 3$, $\log 3$.

10. 2.

11. $x = 1, -\frac{1}{2}$; $y = 2, \frac{1}{2}$.

12. (1) $\dfrac{\log b}{m \log a}$. (2) $\dfrac{\log c}{\log a + \log b}$. (3) $\dfrac{2 \log c - \log a - \log b}{\log a - \log b - \log d}$.

(4) $5 + 2\left(\dfrac{\log 5}{\log 2}\right)$. (5) -1. (6) $\dfrac{\log(\log a) - \log(\log b)}{\log a - \log b}$.

13. 4 and 5, 3 and 4, 1 and 2, 0 and 1, -1 and -2, -3 and -4.

14. 26.

15. 2, $2\frac{1}{2}$, $-\frac{1}{2}$, -1, $\frac{3}{4}$.

16. $\dfrac{x}{n}$.

17. $\sqrt[n]{N}$.

20. $a^{\log bc}$, $a^{\log b}$, $a^{\log \frac{b}{c}}$.

22. $n \log a + \dfrac{n(n-1)}{2} \log r$.

25. $\log b \log m = \log a \log n$.

32. 0, 0.

33. $\log \tan 2A$.

EXERCISE XXIII. (PAGE 168.)

1. $\bar{3}.99780$.
2. $\bar{4}.81875$, $\bar{4}.09868$.
3. .04515.
4. 12 11730, $\overline{12}.83052$.
5. $\bar{1}.44896$, .41328.
6. $\bar{1}.74328$, .50038.
7. 3, 0, -4, -3, -1, 0.
8. .9332848, 5.9332848, $\bar{3}.9332848$
9. 857.6, .08576, 85760000.
10. 486.
11. (1) 4.45495. (2) .58841. (3) $\bar{2}.79449$. (4) $\bar{3}.87506$. (5) .14833. (6) .24857. (7) $\bar{1}.99102$. (8) 1.79931.
12. (1) 14. (2) .17641. (3) 15149. (4) .1001.
13. Half given logs.; half given logs taken negatively.
14. 43.
15. 53855.
16. 9030900, $\bar{1}.6989700$, .6980700, 3.9030900, .1003433, $\bar{2}.4948500$, $\bar{1}.5785580$, .3835050.
17. 1.4771213, .9771213, $\bar{1}.4771213$, $\bar{1}.5228787$, $\bar{2}.0457574$, .2771213, .7276765.
18. .3802112, .1583625, .2676061, 4.4369740, $\bar{1}.5658959$.
19. .7011513, .3360548, $\bar{2}.9654119$.
20. 151.9364.
21. $(1\frac{1}{3})^{15}$.
22. 1.9129.
23. 1.9424.
24. -3.2229.
25. (1) 3.544. (2) 3.881. (3) 17.917. (4) .3716. (5) .2528. (6) 1.257.
26. 20.15 years.
27. 242.7 years.
28. 6377.1 miles.
29. -5.963.
30. $x = .8796$, $y = .2628$.
31. $x = -10.736$, $y = -19.017$.
33. 2.5469, 1.4307.
35. 2.80735.
36. 3.58497.
37. 36.554.
38. .1023.

EXERCISE XXIV. (PAGE 176.)

1. (1) 9.59646. (2) 9.96642. (3) 9.88390.
(4) 10.49273. (5) 10.07312. (6) 10.49654.

2. (1) $15° 23' 9''$. (2) $44° 5' 44''$. (3) $71° 14' 11''$.
(4) $65° 43' 56''$. (5) $32° 17'$. (6) $32° 53'$.

3. 9.8048933, 9.8049583. 4. 9.9691068, 9.9690952.

5. $32° 32' 11''.5$, $57° 27' 43''.7$.

6. $68° 38' 35''.3$, $21° 21' 43''.2$.

9. 9.4225862, 10.0157648, 9.9842346. 10. $79° 44' 50''.4$.

13. (1) $a = 135.17$, $b = 410.31$, $B = 71° 46'$.
(2) $a = 1115.4$, $b = 527.77$, $A = 64° 40' 47''$.
(3) $a = 926.77$, $A = 75° 28' 53''$, $B = 14° 31' 7''$.
(4) $a = 325.64$, $A = 32° 58' 54''$, $B = 57° 1' 6''$.
(5) $b = 122.17$, $c = 149.6$, $A = 54° 45'$.
(6) $a = 1140.7$, $c = 1197.6$, $B = 17° 44' 40''$.
(7) $c = 76.828$, $A = 78° 52' 25''$, $B = 11° 7' 35''$.
(8) $c = 401.53$, $A = 77° 6' 11''$, $B = 12° 53' 49''$.

14. 237.27. 15. 10.493, 7.6237. 16. 212.1.

17. 732.22. 18. 537.19 yards. 19. 178.14.

20. 1000 feet nearly.

EXERCISE XXV. (PAGE 185.)

1. $48° 11' 23''$, $58° 24' 42''$, $73° 23' 55''$.

2. $52° 54' 5''$, $59° 7' 4''$, $67° 58' 51''$.

3. $28° 2' 39''$, $66° 2' 35''$, $85° 54' 46''$.

4. $C = 100° 22' 45''$, $b = 1337.2$, $c = 1758.8$.

5. $B = 118° 53' 34''$, $C = 11° 6' 26''$, $a = 6330.6$.

6. $A = 39°$, $a = 614.44$, $b = 793.69$.

7. $B = 53° 59' 3''$, $C = 76° 44' 57''$, $a = 331.63$.

8. $A = 157° 3' 31''$, $C = 7° 43' 15''$, $b = 105.41$.

9. $B = 54° 10' 56''$, or $125° 49' 4''$.
$C = 73° 30' 4''$, or $1° 51' 56''$, $c = 393.75$, or 13.367.

10. $B = 144° \; 34' \; 45''$, or $18° \; 46' \; 5''$.
$C = 27° \; 5' \; 40''$, or $152° \; 54' \; 20''$, $b = 1590.1$, or 882.72.

11. $B = 71° \; 46'$, $C = 90°$, $b = 410.31$.

12. Impossible. 13. $B = 18° \; 46' \; 5''$, $A = 8° \; 19' \; 35''$, $b = 882.72$.

14. $B = 126° \; 52' \; 11''$, $C = 36° \; 52' \; 11''$.

15. Impossible. 16. 918.02. 17. 731.335, 186.685.

18. 9149.2. 19. $79° \; 6' \; 24''$, $40° \; 53, \; 36''$, 16.414.

20. $71° \; 33' \; 53''$, $61° \; 55' \; 39''$.

21. $A = 5° \; 55' \; 57''$, $B = 28° \; 51' \; 16''$, $C = 145° \; 12' \; 47''$.
$c = 207.22$. $b = 31.754$.

22. 1741.1. 23. 1845.2.

24. $r = 8.8334$, $r_1 = 15.668$, $r_2 = 29.615$, $r_3 = 64.043$, $R = 25.122$.

25. 279.11, 428.7, 442.41.

26. 121.29, 186.29, 192.25. 27. 9, 10, 11.

EXERCISE XXVII. (PAGE 202.)

5. $85° \; 27' \; 33''$, 15.047. 28. $90° - \frac{A}{2}$, $90° - \frac{B}{2}$, $90° - \frac{C}{2}$.

EXERCISE XXVIII. (PAGE 210)

1. $\sqrt{ab}$. 2. 200 feet. 7. 107 feet nearly.

8. 34.42, or 14.52.

22. With the notation of Art. 216 we find,

$$C = 55° \; 58' \; 5'', \; \phi = 49° \; 41' \; 55'', \; \frac{y - x}{z} = 6° \; 16' \; 32'',$$

$$PA = 80.91, \quad PB = 43.054, \quad PD = 109.98.$$

23. In this case the point P is within the triangle ABC. If angles PAB, PCB be denoted by y and x, we shall find

$$\phi = 32° \; 7' \; 50'', \; \frac{y - x}{2} = 16° \; 28' \; 46''.$$

$$PA = 23.656, \quad PB = 58.74, \quad PC = 23.347.$$

EXERCISE XXIX. (PAGE 222.)

9. $1° \; 42'$. 10. 8448 miles. 12. $\frac{2160}{\pi}$ inches. 13. $\frac{a}{b}$.

EXAMINATION PAPERS. (PAGE 224).

PAPER I.

2. $7\frac{1}{2}$ min. 4. $0°, 45°, 180°$. 7. $\frac{19}{32}$. 8. $-\frac{1}{2}\log 2$.

PAPER II.

2. $40°, 60°, 80°$. 4. 0. 6. $A = (n + \frac{1}{4})\pi$.
7. 2. 8. $30°, 60°, 90°$. 9. 28.

PAPER III.

2. 30. 4. $\pm\frac{1}{\sqrt{2}}$, 0. 7. $3\sqrt{21}, \frac{3}{17}\sqrt{21}$. 8. $5\sqrt{31}, 5\sqrt{91}$.

PAPER IV.

2. $65° 24' 30''$. 3. 0. 5. $12\sqrt{4+2\sqrt{2}}$.
7. $n\pi \pm \frac{\pi}{3}$, or $n\pi \pm \frac{\pi}{6}$. 9. $\frac{1}{\log 45 - \log 8}$.

PAPER V.

8. $\cot c = \frac{1}{a} - \frac{1}{b}$. 10. $p = c\sin^3\theta,\ q = c\cos^3\theta$.

PAPER VI.

2. 9 min. 33 sec. 4. $2\theta = 2n\pi \pm \frac{5\pi}{12}$. 6. $n\pi$, or $n\pi \pm \frac{\pi}{3}$.
7. 50 $(\tan 36° + \sin 36°)$ sq. in.
9. $-\frac{1}{2}\sqrt{2-\sqrt{2}},\ -\frac{1}{2}\sqrt{2+\sqrt{2}}$. 10. $2\sin^{-1}\frac{3}{10}, \sin^{-1}\frac{3}{5}$,

PAPER VII.

5. 8.834 min. after 3. 7. $\frac{a^2 \sin B \sin C}{2\sin(B+C)}$.
9. $\bar{8}$ 14709, 2.34797.

PAPER VIII.

2. $a\sqrt{3}$, $\frac{2}{3}a\sqrt{3}$, $\frac{1}{3}a\sqrt{3}$. 6. $p(n^2 - 2m^2) = qmn$.

9. $2\theta = n\pi + (-1)^n \frac{\pi}{3}$, $2\phi = (4n+1)\frac{\pi}{2}$.

PAPER IX.

6. $x^2 + y^2 - 2cy \cot\phi = c^2$.

PAPER X.

1. 8. 2. 4.1645, $-.774$. 4. $\tan\frac{A}{2}\cos A$, $\bar{1}.47713$.

5. $C = 29^\circ\ 57'\ 30''$, $A = 26^\circ\ 22'\ 30''$.

6. $B = 33^\circ\ 40'$, $C = 33^\circ\ 40'$, $a = 355.22$. 7. 5162.3.

8. $19^\circ\ 28'\ 17''$. 9. $26^\circ\ 33'\ 54''$. 10. $75(\sqrt{3}+1)$.

11. $52^\circ\ 15'\ 37''$.

PAPER XI.

1. 200. 2. .50126. 3. .000078492.

4. 8.64. 5. 9.62193, 10.37791, $65^\circ\ 14'\ 20''$.

6. $A = 109^\circ\ 47'$, $B = 18^\circ\ 13'$.

7. $a = 30$, $b = 133$, $c = 136.3$, $B = 77^\circ\ 17'$. 8. $7^\circ\ 12'$.

PAPER XII.

1. $A = 118^\circ$, $B = 82^\circ$, $C = 70^\circ$.

3. $\frac{3}{10}\sqrt{11}$, $\frac{1}{33}\sqrt{11}$, $3\sqrt{11}$, $\frac{10}{33}\sqrt{11}$, $\frac{3}{50}\sqrt{11}$, $\frac{49}{50}$.

4. .994. 7. (1) $\frac{\cos(x+y)}{\cos x \cos y}$, (2) $\tan x \tan y$, (3) $\tan^2 x$.

8. (5) $A = 120^\circ$, $B = C = 30^\circ$.

11. (1) $A = 75^\circ\ 25'\ 54''.7$, $B = 59^\circ\ 27'\ 5''.3$.
(2) $A = 18^\circ\ 23'\ 12''.4$, $C = 111^\circ\ 36'\ 47''.6$.